ADM Energy, Linear Momentum And Angular Momentum In Asymptotically Flat Spacetime

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1 Introduction

Noether's theorem states that any continuous symmetry is associated to a conserved quantity. In general relativity, however, if we defined the symmetry to be the symmetry of gravitational field g itself, we will have two problems. First, it is very hard to find a Killing vector field for the metric g and g itself may not have Killing vector field, thus we may not be able to define conserved quantity for arbitrary spacetime. Second, in classical mechanics, energy corresponds to time translation symmetry. In general relativity, there is an ambiguity in defining time translation as there is no preferred time function, which makes it hard to define energy for general spacetime. In this paper, we focus on asymptotically flat spacetime and derive from Noether's theorem the defining equation for ADM energy, linear momentum and angular momentum. The main references for this paper are [1] and [4]. The precise definitions of the related geometrical concepts used in this paper can be found in [2] and [3].

2 Asymptotic Flatness

2.1 Local Time Function And Its Lapse Function

In this section we introduce some basic concepts in general relativity.

Definition 2.1. *A spacetime is a pair (N, g) where N is an oriented differentiable 4-manifold and g be Lorentzian metric on N.*

Definition 2.2. *A spacelike hypersurface Σ of spacetime (N, g) is a 3-submanifold of (N, g) such that $(\Sigma, g|_\Sigma)$ is a Riemannian manifold.*

Definition 2.3. *Let $\mathcal{U}$ be an open subset of N and $t : \mathcal{U} \to \mathbb{R}$ be a smooth function. t is called a local time function on $\mathcal{U}$ if $dt \cdot X_p > 0$, for all future oriented timelike vector $X_p \in T_p\mathcal{U}$ and all $p \in \mathcal{U}$.*

From Definition 2.3 we know that any $s \in \mathbb{R}$ is a regular value of the function t. Hence, $\Sigma_s \equiv t^{-1}(s)$ is a submanifold of $\mathcal{U}$ and $\{\Sigma_s\}_{s\in\mathbb{R}}$ forms a foliation of $\mathcal{U}$.

Definition 2.4. *Let (y^0, y^1, y^2, y^3) be a chart on an open subset W of $\mathcal{U}$. The function*

$$\Phi = (-g^{\mu\nu}\partial_\mu t\partial_\nu t)^{-1/2}. \tag{1}$$

is called the lapse function corresponding to the local time function t.

Note that if $dt \cdot X_p > 0$ for all timelike vector $X \in T_p\mathcal{U}$ and all $p \in \mathcal{U}$, then $g^{\mu\nu}\partial_\mu t\partial_\nu t < 0$ and the lapse function is well defined and positive for all $p \in \mathbb{U}$ is independent of the choice of coordinate system around $p \in \mathbb{U}$. The physical meaning of the lapse function is the following: think of a free falling clock in $\mathcal{U}$, suppose the clock measures proper time $\delta\tau$ after it moves from the leaf Σ_t to $\Sigma_{t+\delta t}$, then we have

$$\Phi = \lim_{\delta\tau\to 0} \frac{\delta\tau}{\delta t}. \tag{2}$$

In other words, the lapse function is the ratio between proper time and coordinate time.

2.2 Second Fundamental Form

Let (N, g) be a spacetime and $\mathcal{H} \subset N$ be spacelike hypersurface. At each point $x \in \mathcal{H}$, we can find a future directed timelike vector $N_x \in T_xN$ such that

$$\begin{aligned} &g(N_x, X_x) = 0, \forall X_x \in T_xH, \\ &g(N_x, N_x) = -1. \end{aligned} \tag{3}$$

This N_x is called future directed unit normal to $\mathcal{H}$. In this way, we can assign a vector field N to $\mathcal{H}$.

Definition 2.5. *The second fundamental form k of a spacelike hypersurface $\mathcal{H} \subset N$ is a 2-covariant symmetric tensor field on $\mathcal{H}$ whose value at $x \in \mathcal{H}$ is defined by*

$$k(X, Y) = g(\nabla_X N, Y), \forall X, Y \in T_x\mathcal{H}. \tag{4}$$

The second fundamental form is a measurement of how the vector field N changes its direction. Hence it is a measure of how "curved" the hypersurface $\mathcal{H}$ is as viewed in N.

2.3 Asymptotically Flat Data Set

Definition 2.6. *Let $(\mathcal{H}, \bar{g}, k)$ be a triple where $\mathcal{H}$ is a 3-manifold, $\bar{g}$ is a Riemannian metric on $\mathcal{H}$ and k is a covariant symmetric 2-tensor field on $\mathcal{H}$. The triple is said to be an asymptotically flat data set if*
1. There exists a compact set $\mathcal{K} \subset \mathcal{H}$ such that $\mathcal{H} \backslash \mathcal{K}$ is diffeomorphic to $\mathbb{R}^3 \backslash \mathcal{B}$, where $\mathcal{B}$ is the unit ball in $\mathbb{R}^3$.
2. There exists a chart (y^1, y^2, y^3) covering $\mathcal{H} \backslash \mathcal{K}$ relative to which $\bar{g}_{ij} \to \delta_{ij}$, $k_{ij} \to 0$ as $r = \sum_{i=1}^{3} (y^i)^2 \to \infty$.

In the general relativity setup, we should think of $\mathcal{H}$ as a spacelike submanifold of (N, g) with $\bar{g}$ the induced metric and k the second fundamental form. If there is a global time function t on N, the physical situation for asymptotically flat data set is the following. At each time $t = s$, we have a spacelike hypersurface $\mathcal{H} = t^{-1}(s)$. The matter field is only contain in a compact (finite) region of $\mathcal{H}$. With this physical picture in mind, we make the following assumption on the spacetime (N, g):

Assumption: There exists a function t on N and a closed set $W \subset N$ such that:
1. The range of s is $\mathbb{R}$.
2. $t^{-1}(s) \cap W$ is compact.
3. $t^{-1}(s) \cap (N \backslash W)$ is a spacelike hypersurface diffeomorphic to $\mathbb{R}^3 \backslash \mathcal{B}$, where $\mathcal{B}$ is the unit ball in $\mathbb{R}^3$.

Here, we can think of the function t on N as a global time function on $N \backslash W$ and $N \backslash W$ as the region where there is no matter field. The reason to remove the closed set W in N is the following. There can be spacetime singularities (say, black holes) in W and W can have complicated topological structure. However, we only care about the total energy of the entire system at a given time slice, which is a property of the system at the infinity. Now let $M = N \backslash W$, $\Sigma_s = t^{-1}(s) \cap M$. In the following part of the paper, we mainly focus on (M, g), the sub-spacetime where there is no matter field.

3 The Lagrangian Formalism Of General Relativity

3.1 The Background Metric

Since M is diffeomorphic to $(\mathbb{R}^3 \backslash \mathcal{B}) \times \mathbb{R}$, we can find a single chart ψ covering M, whose coordinate is denoted as (x^0, x^1, x^2, x^3) such that $x^0(p) = t(p)$ for all $p \in M$. With this coordinate system, we can assign a 2-covariant symmetric tensor field η on M whose coordinate representation under (x^0, x^1, x^2, x^3) is

$$\eta_{\mu\nu} = \mathrm{diag}\{-1, 1, 1, 1\}. \tag{5}$$

This metric η is called the background metric. Let ∇ be the metric connection of η and $d\mu_\eta$ be the volume form associated to η. The physical reason why we need the background metric is the following. The absolute energy doesn't make sense in physics. When physicists talk about energy, they really mean the energy difference between the state they are talking about and a reference state. In general relativity, the reference state is the Minkowski spacetime and we set it energy to be zero. The background metric is used as a reference state on M.

3.2 Configuration Space And Velocity Space

Definition 3.1. *$\mathcal{C} = \cup_{x \in M} L_x M$ is called the configuration space, where $L_x M$ is the open subset of $S_{2x} M$ consisting of quadratic forms on $T_x M$ of index 1.*

Here $\mathcal{C}$ is the space of all possible gravitational fields on M.

Definition 3.2. *$\mathcal{V} = \cup_{x \in M} \mathcal{L}(T_x M, S_{2x} M)$ is called the velocity bundle, where $\mathcal{L}(T_x M, S_{2x} M)$ is the vector space of linear maps from $T_x M$ to $S_{2x} M$.*

The reason why $\mathcal{V}$ is called the velocity bundle is the following: Each $v_x \in \mathcal{L}(T_x M, S_{2x} M)$ specifies a change of gravitational field in the $X \in T_x M$ direction, this is what velocity mean in physics.

3.3 The Einstein-Weyl Lagrangian

Before introducing the Einstein-Weyl Lagrangian, we introduce some operations on $\mathcal{C}$ and $\mathcal{V}$.

Definition 3.3. *The bundle product of $\mathcal{C}$ and $\mathcal{V}$ is the bundle*

$$\mathcal{C} \times_M \mathcal{V} = \cup_{x \in M} L_x M \times \mathcal{L}(T_x M, S_{2x} M). \tag{6}$$

$\mathcal{C} \times_M \mathcal{V}$ corresponds to the phase space in classical mechanics. Let $n = 4$ and $\wedge_n M$ be the bundle of top degree forms on M. Since we have a projection

$$\pi_{\mathcal{C}\times_M\mathcal{V}} : \mathcal{C} \times_M \mathcal{V} \to M, \tag{7}$$

we can define the pullback bundle of $\wedge_n M$ by $\pi_{\mathcal{C}\times_M\mathcal{V}}$.

Definition 3.4. *The pullback bundle of $\wedge_n M$ by $\pi_{\mathcal{C}\times_M\mathcal{V}}$ is defined as*

$$\pi^*_{\mathcal{C}\times_M\mathcal{V}} \wedge_n M = \{(y, \omega_x) | \, \omega_x \in \wedge_{nx} M, y \in \mathcal{C} \times_M \mathcal{V}, \pi_{\mathcal{C}\times_M\mathcal{V}}(y) = x\}. \tag{8}$$

In other words, we put the vector space $\wedge_{nx} M$ on $y \in \mathcal{C} \times_M \mathcal{V}$ and "glue" them together to form a vector bundle. The details of the pullback bundle can be found in [3].

Definition 3.5. *A section $L : \mathcal{C} \times_M \mathcal{V} \to \pi^*_{\mathcal{C}\times_M\mathcal{V}} \wedge_n M$ is called a Lagrangian.*

One advantage of defining Lagrangian as a section is that this definition is independent of coordinate systems and hence satisfies general covariance (invariant under coordinate transformation). The other advantage is that the action arises naturally from this definition. Let $g : M \to \mathcal{C}$ be a section of $\mathcal{C}$. Since we already have a connection ∇ on M, this section g of $\mathcal{C}$ induces a section ∇g of $\mathcal{V}$, defined by

$$\begin{aligned} (\nabla g)_x : T_x M &\to S_{2x} M \\ Y &\mapsto \nabla_Y g, \end{aligned} \tag{9}$$

In other words, any section g of $\mathcal{C}$ induces a section $(g, \nabla g)$ of $\mathcal{C} \times_M \mathcal{V}$. Composing the section with L we obtain a section $L(g, \nabla g)$ of $\wedge_n M$.

Definition 3.6. *The functional $g \to \int_M L(g, \nabla g)$ is called the action of the Lagrangian L.*

Since g is a section of $\mathcal{C}$, we can define its Einstein tensor G.

Definition 3.7. *A Lagrangian L is called a Lagrangian for the gravitational field if*

$$\frac{\delta \int_M L(g, \nabla g)}{\delta g} = G. \tag{10}$$

Here $\frac{\delta \int_M L(g,\nabla g)}{\delta g}$ is a functional derivation [4]. Note that the Euler-Lagrangian equation for L is defined to be the equation

$$\frac{\delta \int_M L(g, \nabla g)}{\delta g} = 0. \tag{11}$$

Hence if L is a Lagrangian for the gravitational field, its Euler-Lagrangian gives the vacuum Einstein equation and this is where its name comes from.

Theorem 3.1. *Let $\Gamma^\alpha_{\beta\kappa}$ be the Christoffel symbol of a metric g, then the Lagrangian*

$$L = L^* d\mu_\eta, \tag{12}$$

is a Lagrangian of the gravitational field called the Einstein-Weyl Lagrangian, where L^* is represented under the x^μ coordinate as

$$L^* = -\frac{1}{4}\sqrt{-g} g^{\mu\nu} (\Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\nu\beta} - \Gamma^\beta_{\mu\nu}\Gamma^\alpha_{\beta\alpha}). \tag{13}$$

Physicists might be more familiar with the Hilbert Lagrangian $L_H = -\frac{1}{4} R_g d\mu_g$, where R_g is the Ricci scalar of g and $d\mu_g$ is the volume form associated to g. However, the formulation we developed above only works for Lagrangian that contains only first order derivatives of the unknown metric and the Hilbert Lagrangian contains second order derivatives of the unknown metric. This is why we introduce the Einstein-Weyl Lagrangian. It is worth noting that first, under the x^μ coordinate system, the Einstein-Weyl Lagrangian differs by a divergence $\nabla_\mu I^\mu$ from the Hilbert action. Hence their Euler-Lagrangian equation all give the Einstein vacuum equation. And second, L_H is gauge invariant (to be explained in Subsection 5.1) in the sense that for all diffeomorphism $f : M \to M$ and all section g of $\mathcal{C}$ we have

$$f^* L_H = -\frac{1}{4} R_{f^*g} d\mu_{f^*g}. \tag{14}$$

While the Einstein-Weyl Lagrangian is not gauge invariant, i.e. there exists a diffeomorphism f on M and a section g of $\mathcal{C}$ such that

$$f^*(L(g, \nabla g)) \neq L(f^*g, \nabla f^*g). \tag{15}$$

For any top degree form ω we have

$$\int_M \omega = \int_M f^*\omega, \tag{16}$$

whenever $f : M \to M$ is a diffeomorphism. Eq. (14) implies the identity

$$\int_M -\frac{1}{4} R_g d\mu_g = \int_M -\frac{1}{4} R_{f^*g} d\mu_{f^*g}, \tag{17}$$

i.e., the action of the Hilbert Lagrangian is invariant under any gauge transformation $g \mapsto f^*g$ (the reason why this is called a gauge transformation is explained in Subsection 5.1). In contrast, the action of the Einstein-Weyl Lagrangian may not be invariant under some gauge transformation.

4 Noether's Theorem

4.1 Lie Derivatives

Let $f : M \to M$ be a diffeomorphism. We extend the action of f to $\mathcal{C}$, $\mathcal{V}$ and a general Lagrangian $L = L^* d\mu_\eta$ in the following way.

Definition 4.1. *The action of f on $\mathcal{C}$ is defined by*

$$q \in \mathcal{C}_x \mapsto f_* q \in \mathcal{C}_{f(x)}, x \in M, \tag{18}$$

where for all $Y_1, Y_2 \in T_{f(x)}M$ we have

$$(f_* q)(Y_1, Y_2) = q(df^{-1} \cdot Y_2, df^{-1} \cdot Y_2) \tag{19}$$

Definition 4.2. *The action of f on $\mathcal{V}$ is defined by*

$$v \in \mathcal{V}_x = \mathcal{L}(T_x M, S_{2x} M) \mapsto f_* v \in \mathcal{V}_{f(x)}, x \in M, \tag{20}$$

where for all $Y \in T_{f(x)}M$ we have

$$(f_* v)(Y) = f_*(v(df^{-1} \cdot Y)). \tag{21}$$

Definition 4.3. *The pullback of L by f is defined to be f^*L, where*

$$(f^*L)(q, v)(Y_1, ..., Y_n) = L(f_* q, f_* v)(df \cdot Y_1, ..., df \cdot Y_n), \tag{22}$$

for all $Y_1, ..., Y_n \in T_x M$.

Now now define the Lie derivative of L with respect to a vector field X on M.

Definition 4.4. *Let X be a vector field on M whose flow is the 1-parameter group $\{f_t\}$ of diffeomorphism of M onto itself, the Lie derivative of L by X is defined to be*

$$\mathcal{L}_X L \equiv \frac{d(f_t^* L)}{dt} \tag{23}$$

4.2 Noether Current

The coordinate system x^μ induces a coordinate system $q_{\alpha\beta}$ on $\mathcal{C}$ and $v_{\mu\alpha\beta}$ on $\mathcal{V}$ by

$$\begin{aligned} q &= q_{\alpha\beta} dx^\alpha \otimes dx^\beta, q \in \mathcal{C}_x, \\ v &= v_{\mu\alpha\beta} dx^\mu \otimes dx^\alpha \otimes dx^\beta, v \in \mathcal{V}_x. \end{aligned} \tag{24}$$

Definition 4.5. *The canonical momentum of a general Lagrangian $L = L^* d\mu_\eta$ under coordinate system x^μ is defined as*

$$p^{*\mu\alpha\beta} = \frac{\partial L^*}{\partial v_{\mu\alpha\beta}}. \tag{25}$$

Definition 4.6. *A current J is a section of $\pi^*_{\mathcal{C}\times_M \mathcal{V}} \wedge_{n-1} M$*

Since we already have a volume form $d\mu_\eta$ on M, we can identify J with a section J^* of $\pi^*_{\mathcal{C}\times_M\mathcal{V}}TM$ by

$$J(q,v) = i_{J^*(q,v)}d\mu_\eta, \tag{26}$$

where $(q,v)\in\mathcal{C}_x\times\mathcal{V}_x$ and $i_{J^*(q,v)}$ is the interior multiplication by $J^*(q,v)$ defined by

$$(i_{J^*(q,v)}d\mu_\eta)(Y_1,...,Y_{n-1}) = d\mu_\eta(J^*(q,v),Y_1,...,Y_{n-1}), \tag{27}$$

where $Y_1,...,Y_{n-1}\in T_xM$. Now, we define Noether current.

Definition 4.7. *Let X be a vector field on M, the Noether current corresponding to X is a current J_X whose representation under x^μ coordinate is*

$$J_X^{*\mu}(q,v) = p^{*\mu\alpha\beta}(X^\nu v_{\nu\alpha\beta} + q_{\nu\beta}\nabla_\alpha X^\nu + q_{\alpha\nu}\nabla_\beta X^\nu) - L^*X^\mu, \tag{28}$$

where $J_X^{*\mu}$ is the identification of J_X under $d\mu_\eta$.

4.3 Noether's Theorem And Local Conservation Law

With all the terminology defined above, we state Noether's theorem.

Theorem 4.1. *Let X be a vector field on M, $L = L^*d\mu_\eta$ be a general Lagrangian, J_X be the of L corresponding current of X. Then for every solution g of the Euler-Lagrangian equation we have*

$$d(J_X\circ(g,\nabla g)) = K\circ(g,\nabla g), \tag{29}$$

where under the x^μ coordinates, we have

$$\begin{aligned} &K = -\mathcal{L}_X L - T,\\ &T^{*\mu} = -p^{*\mu\alpha\beta}\{q_{\nu\beta}(\mathcal{L}_X\nabla)^\nu_{\mu\alpha} + q_{\alpha\nu}(\mathcal{L}_X\nabla)^\nu_{\mu\beta}\},\\ &(\mathcal{L}_X\nabla)(Y,Z) = [X,\nabla_Y Z] - \nabla_{[X,Y]}Z - \nabla_Y[X,Z]. \end{aligned} \tag{30}$$

Theorem 4.1 is a general statement true for all Lagrangian and all vector fields. Now we consider Einstein-Weyl Lagrangian and let X be a vector field that generates a symmetry of the system.

Corollary 4.1.1. *If X is a Killing vector field of η and L is the Einstein-Weyl Lagrangian, we have*

$$\mathcal{L}_X L = 0,\ \mathcal{L}_X\nabla = 0. \tag{31}$$

In particular, we have local conservation of the Noether's current

$$d(J_X\circ(g,\nabla g)) = 0, \tag{32}$$

for every solution g of the Euler-Lagrangian equation.

Here we comment on why Eq. (32) is called a local conservation law. Let $J_X\circ(g,\nabla g) = \alpha$ and write

$$\alpha = \alpha_0 dx^1\wedge dx^2\wedge dx^3 + \alpha_1 dx^0\wedge dx^2\wedge dx^3 + \alpha_2 dx^0\wedge dx^1\wedge dx^3 + \alpha_3 dx^0\wedge dx^1\wedge dx^2 \tag{33}$$

For any point $p\in M$, let (z^0,z^1,z^2,z^3) be the coordinate of p and choose r, δ_1 and δ_2 sufficiently small such that the cylinder

$$R = \{(x^0,x^1,x^2,x^3)|\sum_{i=1}^{3}(x^i-z^i)^2\le r^2, \delta_1\le x^0-z^0\le\delta_2\} \tag{34}$$

lies in the image of ψ. Since α is closed, i.e., $d\alpha=0$, we can use Stoke's theorem and obtain

$$0 = \int_{\psi^{-1}R} d\alpha = \int_{\partial(\psi^{-1}R)}\alpha \tag{35}$$

Since the coordinate x^0 aligned with t, i.e., $t(p) = x^0(p)$, $A_{\delta_1} = \Sigma_{z^0-\delta_1}\cap\psi^{-1}R$ and $A_{\delta_2} = \Sigma_{z^0-\delta_2}\cap\psi^{-1}R$ are the bottom and top boundary of $\psi^{-1}R$. Let $B = \cup_{s\in[z^0-\delta_1,z^0+\delta_2]}\Sigma_s\cap\partial(\psi^{-1}R)$, then we have

$$\int_{\partial(\psi^{-1}R)}\alpha = \int_{A_{\delta_1}}\alpha + \int_{A_{\delta_2}}\alpha + \int_B\alpha = 0. \tag{36}$$

From Eq. (33) we know

$$\begin{aligned}\int_{A_{\delta_1}} \alpha &= \int_{A_{\delta_1}} \alpha_0 dx^1 \wedge dx^2 \wedge dx^3, \\ \int_{A_{\delta_2}} \alpha &= -\int_{A_{\delta_2}} \alpha_0 dx^1 \wedge dx^2 \wedge dx^3,\end{aligned} \tag{37}$$

where the minus sign comes from the orientation. Since α is closed and $\psi^{-1}R$ is contractible, from Poincaré lemma we know α is also exact, i.e. $\alpha = d\beta$ for some $n-2$ form β on M. Use Stock's theorem again, we obtain

$$\int_B \alpha = \int_{\partial B} \beta = \int_{\partial A_{\delta_1}} \beta + \int_{\partial A_{\delta_2}} \beta. \tag{38}$$

Expand β in x^μ coordinate, we can write

$$\beta = \beta_{01} dx^2 \wedge dx^3 + \beta_{02} dx^1 \wedge dx^3 + \beta_{03} dx^1 \wedge dx^2 + \beta_{12} dx^0 \wedge dx^3 + \beta_{13} dx^0 \wedge dx^2 + \beta_{23} dx^0 \wedge dx^1. \tag{39}$$

This gives us

$$\begin{aligned}\int_{\partial A_{\delta_1}} \beta &= \int_{\partial A_{\delta_1}} \beta_{03} dx^1 \wedge dx^2 + \beta_{02} dx^1 \wedge dx^3 + \beta_{01} dx^2 \wedge dx^3, \\ \int_{\partial A_{\delta_2}} \beta &= -\int_{\partial A_{\delta_2}} \beta_{03} dx^1 \wedge dx^2 + \beta_{02} dx^1 \wedge dx^3 + \beta_{01} dx^2 \wedge dx^3.\end{aligned} \tag{40}$$

Combining Eq. (35), Eq. (37) and Eq. (40) and use the fact that δ_2 and δ_2 are arbitrary, we see that

$$Q_s = \int_{A_s} \alpha_0 dx^1 \wedge dx^2 \wedge dx^3 + \int_{\partial A_s} \beta_{03} dx^1 \wedge dx^2 + \beta_{02} dx^1 \wedge dx^3 + \beta_{01} dx^2 \wedge dx^3 \tag{41}$$

is independent of s, where $A_s = \Sigma_s \cap \psi^{-1}R$. Physically speaking, the term $\int_{A_s} \alpha_0 dx^1 \wedge dx^2 \wedge dx^3$ corresponds to the amount of charge enclosed by the surface ∂A_s and $\int_{\partial A_s} \beta_{03} dx^1 \wedge dx^2 + \beta_{02} dx^1 \wedge dx^3 + \beta_{01} dx^2 \wedge dx^3$ is the amount of charge flow through ∂A_s into A_s. This is why Eq. (32) is called a conservation law. It is local because Eq. (41) is only true for small region A_s (recall we used the fact that closeness is equivalent to exactness in $\psi^{-1}R$). In order to have a "global" conservation law, i.e., the charge flow out of an arbitrary surface equals the decrease of the amount of charge enclosed by that surface, $J_X \circ (g, \nabla g)$ need to be exact for every solution g of the Euler-Lagrangian equation.

4.4 Motivation From Classical Mechanics

Noether's theorem says every symmetry of a physical system gives rise to a local conservation law. In classical mechanics, the spacetime is $\mathbb{R}^4$, local conservation law is equivalent to global conservation law and gives conserved quantity. The conserved quantity associated to time translation is called energy. The conserved quantity associated to a space translation is called linear momentum. The conserved quantity associated to a space rotation is called angular momentum.

5 ADM Energy, Linear Momentum And Angular Momentum

5.1 Gauge Invariance And Conserved Quantity

Let U be a 4-manifold and g a gravitational field on U. Let $f : U \to U$ be a diffeomorphism, the physical laws on (U, g) and (U, f^*g) are the same, i.e., we can not distinguish (U, g) and (U, f^*g) from any experiment done in these two spaces. The reason is the following, let ϕ be a chart on U, for simplicity we assume this chart covers U. Precomposing ϕ by f we get another chart $f \circ \phi$ on U. Direct computation shows that

$$(f^*g)_{\mu\nu}(p) = g_{\mu\nu}(f(p)). \tag{42}$$

Hence, the physical law around $f(p)$ in (U, g) and p in (U, f^*g) are the same. In other words, we should think f^*g and g as equivalent gravitational field on U. Therefore, energy, linear momentum and angular momentum should be all the same for f^*g and g. This motivates the following definition of gauge invariance.

Definition 5.1. *Let Q be a functional from sections of $\mathcal{C}$ to $\mathbb{R}$. Q is said to be gauge invariant if for all vector field X on M we have*

$$\frac{d(Q(f_t^*g))}{dt} = 0, \tag{43}$$

where $f_t : M \to M$ is the flow of X.

In classical mechanics we define energy, linear momentum and angular momentum in time slice Σ_s. The definition in general relativity should also have such property. For quantities with this property, we have the notion of a conserved quantity.

Definition 5.2. *Let Q_s be quantities associated to the time slice Σ_s, Q_s is said to be conserved if*

$$\frac{dQ_s}{ds} = 0. \tag{44}$$

5.2 Remark: General Covariance, Lorentz Invariance And Gauge Invariance

In physics literature, general covariance, Lorentz covariance and gauge invariance arises repeatedly. However, the physics literature often use coordinate transformation to "representation" these three concepts, which causes confusion for beginners. Here, we try to explain these three concepts from a global point of view.

General covariance refers to the fact that physical laws (the equation of motion) should not depend on coordinate systems. One way to achieve general covariance is to first find a fixed coordinate system, then write a theory in that coordinate system and require the corresponding quantities we write to be a scalar, vector, tensor or spinor (this is what we do in defining the Einstein-Weyl Lagrangian). This is a minimum requirement of physical laws.

Lorentz covariance is a concept in special relativity, which refers to the fact that physical laws have the same form in different inertial frame. We use the following example to show the relation between general covariance and Lorentz covariance. Let (x^0, x^1, x^2, x^3) be the standard chart on $\mathbb{R}^4$ and $\eta_{\mu\nu} = \text{diag}\{-1, 1, 1, 1\}$ be the Minkowski metric on $\mathbb{R}^4$. Fix this coordinate system, consider a scalar field $\phi : \mathbb{R}^4 \to \mathbb{R}$. We can require that in the x^μ coordinate system the physical law (the equation of motion) for ϕ is

$$\frac{\partial \phi}{\partial x^0} = 0, \tag{45}$$

and $\frac{\partial \phi}{\partial x^0}$ is a scalar. Then the physical law determined by Eq. (45) satisfies general covariance. Under a Lorentz transformation $(x^0, x^1, x^2, x^3) \mapsto (y^0, y^1, y^2, y^3)$, the equation of motion transforms into

$$\frac{\partial \phi}{\partial y^\mu} \frac{\partial y^\mu}{\partial x^0} = 0. \tag{46}$$

Since Eq. (45) and Eq. (46) are not of the same form, then the physical law determined by Eq. (45) doesn't satisfy Lorentz covariance. Now, we require that in the x^μ coordinate system a new physical law (the equation of motion) for ϕ is

$$\eta^{\mu\nu} \frac{\partial^2 \phi}{\partial x^\mu x^\nu} = 0, \tag{47}$$

and $\eta^{\mu\nu} \frac{\partial^2 \phi}{\partial x^\mu x^\nu}$ is a scalar. Then the physical law determined by Eq. (47) satisfies general covariance. Under a Lorentz transformation $(x^0, x^1, x^2, x^3) \mapsto (y^0, y^1, y^2, y^3)$, the equation of motion transforms into

$$\eta^{\mu\nu} \frac{\partial^2 \phi}{\partial y^\mu y^\nu} = 0, \tag{48}$$

Since Eq. (47) and Eq. (48) are of the same form, the physical law determined by Eq. (47) satisfies Lorentz covariance. In terms of geometric language, a general covariant physical law ($F(\phi) = 0$) for some field ϕ in special relativity is Lorentz covariant, if for all Killing vector field X of $(\mathbb{R}^4, \eta)$, we have

$$g_t^*(F(\phi)) = F(g_t^*\phi), \tag{49}$$

where g_t is the flow of X. To get the physical meaning of Eq. (49), we introduce the notion of an observer.

Definition 5.3. *Let (N, g) be a spacetime. An observer a pair (N_i, f_i) with $f_i : N_i \to N$ being a diffeomorphism. The observed quantity by the observer is the pullback of the corresponding quantity on (N, g) by f_i.*

From the definition above, it is easy to understand why metric g and f^*g are equivalent: they are the same physical quantity (the gravitational field) observed by different observers. Now we look at Eq. (49). Let i and j be two observers whose transition function $f_j^{-1} \circ f_i : N_i \to N_j$ is given by f_t. Now we assume that i is in a inertial frame. Since f_t is the flow of the Killing vector field X of η, we know that if i is in a inertial frame, so is j. Let ϕ be some field observed by j. Then $f_t^*\phi$ is same field observed by i. Let $F(\phi)$ be a field which is a consequence of ϕ observed by j. Then $F(f_t^*\phi)$ is a fields which is a consequence of $f_t^*\phi$ observed by i. The requirement that physical laws observed by i and j are the same implies $F(f_t^*\phi) = f_t^*(F(\phi))$. In other words, Eq. (49) is a restatement that physical laws observed by different inertial observer are the same. Hence, Lorentz covariance is a stronger requirement than general covariance. Gauge invariance refers to the fact that physical observables should take the same value on physically equivalent fields. In the general relativity setting, this is a stronger requirement than Lorentz Invariance.

5.3 Exactness Of Noether Current

Noether's theorem states that to each symmetry of the physical system we can associate a local conserved quantity. However, in order to define a conserved quantity for the entire system, we must show that $J_X(g, \nabla g)$ is exact for all Killing vector field X and all solution g of the Euler-Lagrangian equation. Indeed, we have the following Proposition.

Proposition 5.1. *Let X be a Killing vector field of (M, η), there exists a section G_X of $\pi^*_{\mathcal{C}\times_M \mathcal{V}} \wedge_{n-2} M$ such that for every solution g of the Euler-Lagrangian equation we have*

$$J_X \circ (g, \nabla g) = d(G_X \circ (g, \nabla g)) \tag{50}$$

Proof. See [1] page 50 to 52. □

5.4 Quantity Associated To A Killing Vector Field

The discussion in Subsection 4.3 motivate us to define a conserved quantity associated to Killing vector field X of η to be something like $\int_{\Sigma_s} J_X \circ (g, \nabla g)$. This, however, has the drawback that it is related to gravitational field near the matter field and integration on an non-compact manifold is not always well-defined. To resolve the first problem, note that $J_X \circ (g, \nabla g) = d(G_X \circ (g, \nabla g))$, Stoke's theorem then tells us $\int_B J_X \circ (g, \nabla g) = \int_{\partial B} G_X \circ (g, \nabla g)$. This motivates the following definition.

Definition 5.4. *Let S be a closed 2-surface in M that is diffeomorphic to S^2, and X be a Killing vector field of η, g be a solution of Einstein vacuum equation on M. The quantity associated to (X, S, g) is defined as*

$$Q_X(S, g) = \int_S G_X(g, \nabla g). \tag{51}$$

The surface should eventually enclose all the gravitation field in a time slide motivates the following definition.

Definition 5.5. *An exhaustion of Σ_s is a sequence $\{B_{ns} |\, n \in \mathbb{N}^+\}$ of open set in Σ_s such that the boundary of B_{ns} is diffeomorphic to S^2 and $\cup_{n=1}^{\infty} B_{ns} = \Sigma_s$.*

Now we are ready to define conserved quantity associated to Killing vector field X.

Definition 5.6. *Let $\{B_{ns}\}$ be an exhaustion of Σ_s, X be a Killing vector field of η, g be a metric on N solving the Einstein equation (Hence solve the vacuum Einstein equation on M). We define the quantity associated to X to be*

$$Q_X(g, s) \equiv \lim_{n\to\infty} \int_{\partial B_{ns}} G_X(g, \nabla g). \tag{52}$$

5.5 ADM Energy and Linear Momentum: Gauge Invariance And Conservation

Let $X_{(0)} = \frac{\partial}{\partial x^0}$ be the Killing vector field generating time translation and $X_{(i)} = \frac{\partial}{\partial x^i}, i = 1, 2, 3$ be the Killing vector field generating spatial translation along the i^{th} spatial direction.

Definition 5.7. *The AMD energy is defined to be*

$$E(g, s) = Q_{X_{(0)}}(g, s). \tag{53}$$

The AMD linear momentum along the i^{th} spatial direction is defined to be

$$P_i(g, s) = Q_{X_{(i)}}(g, s). \tag{54}$$

In certain asymptotically flat spacetime, the ADM energy and AMD linear momentum are well-defined, gauge invariant and conserved.

Definition 5.8. *A function $h(r)$ is said to be $o_l(r^{-\alpha})$ as $r \to \infty$ if $h(r)$ has continuous derivative up to l^{th} order and $h^{(m)}(r)$ (the m^{th} derivative of h) is of $o(r^{-\alpha-m})$.*

Theorem 5.1. *If for any $s \in \mathbb{R}$, we can find a chart (y^1, y^2, y^3) covering a neighborhood of infinity around Σ_s such that $g_{0i} = 0$ on the hypersurface Σ_s, and*

$$\begin{aligned} \Phi(r) &= 1 + o_2(r^{-\alpha}), \\ \bar{g}_{ij}(r) &= \delta_{ij} + o_2(r^{-\alpha}), \\ k_{ij}(r) &= o_1(r^{-1-\alpha}), \end{aligned} \tag{55}$$

where $r = \sum_{i=1}^{3} (y^i)^2$ and $\alpha > 1/2$. Then $E(g, s)$ and $P_i(s)$ are independent of the exhaustion, gauge invariant and conserved.

Proof. See [1] page 53 to 58. □

The explicit formula for E and P_i are the following.

$$
\begin{aligned}
E &= \frac{1}{4}\lim_{r\to\infty}\int_{S_r}\sqrt{\bar{g}}(\bar{g}^{jm}\bar{g}^{in}-\bar{g}^{ij}\bar{g}^{mn})\nabla_j\bar{g}_{mn}dS_i,\\
P_j &= -\frac{1}{2}\lim_{r\to\infty}\int_{S_r}\sqrt{\bar{g}}(\bar{g}^{im}k_{jm}-\delta^i_j\bar{g}^{mn}k_{mn})dS_i,
\end{aligned}
\tag{56}
$$

where $\bar{g}_{ij}, k_{ij}$ are the induced metric and second fundamental form of Σ_0 under coordinate (x^1, x^2, x^3) and dS_i is the area element on the coordinate sphere S_r.

5.6 ADM Angular Momentum: Gauge Invariance And Conservation

Let $X_{(\alpha\beta)} = x_\alpha\frac{\partial}{\partial x^\beta} - x_\beta\frac{\partial}{\partial x^\alpha}$ be the Killing vector field generating spacetime rotations. Then $X_{(ij)}$ is the space rotation along the k^{th} spatial axis if (i, j, k) is a permutation of $(1, 2, 3)$ and $J_k = Q_{X_{(ij)}}$ is the k^{th} component of the angular momentum.

Theorem 5.2. *If for any $s \in \mathbb{R}$, we can find a chart (y^1, y^2, y^3) covering a neighborhood of infinity around Σ_s such that*

$$
\begin{aligned}
\bar{g}_{ij} &= (1+\frac{2M}{r})\delta_{ij} + \mathcal{O}_2(r^{-1-\epsilon}),\\
k_{ij} &= \mathcal{O}(r^{-2-\epsilon}),
\end{aligned}
\tag{57}
$$

where $\epsilon > 0$. Then the angular momentum component is independent of exhaustion, gauge invariant and conserved.

Proof. See [1] page 70 to 72. □

The explicit formula for the angular momentum component J_k is the following

$$
J_k = -\frac{1}{2}\lim_{r\to\infty}\int_{S_r}\sum_{j=1}^{3}\epsilon_{klj}x^l\sqrt{\bar{g}}(\bar{g}^{im}k_{jm}-\delta^i_j\bar{g}^{mn}k_{mn})dS_i,
\tag{58}
$$

where ϵ_{klj} is the Levi-Civita symbol. The conserved quantity associated to $X_{(0\mu)}$ is called the center of mass integral. under the same fall off rate in Eq. (57), it is independent of exhaustion, gauge invariant and conserved. The proof of this fact can be found in [1] page 72 to 77.

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