Group Theory with Applications in Solid State Physics

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Contents

1 Linear algebra recap

Throughout this note, all vector spaces are finite dimensional over $\mathbb{C}$ and all groups are assumed to be finite. We assume the reader is familiar with the computation matrices. In this section, we reintroduce linear algebra in an "intrinsic" way.

1.1 Matrix representation of linear map

Let V be an n dimensional vector space and $F : V \to V$ be a linear map over V. Given a basis $\{e_i\}_{i=1}^n$ of V, we can associate a matrix $M(F)$ to F in the following way. Since $\{e_i\}_{i=1}^n$ is a basis of V, $F(e_i)$ can be expressed as a linear combination of $\{e_i\}_{i=1}^n$:

$$F(e_i) = \sum_{j=1}^{n} a_{ji} e_j. \tag{1}$$

We defined the matrix $M(F)$ to be

$$M(F)_{ji} = a_{ji} \tag{2}$$

An important property of this association is the following:

Proposition 1.1. *Let $F : V \to V$ and $G : V \to V$ be two linear maps and $\{e_i\}_{i=1}^n$ be a basis of V, $M(F)$, $M(G)$ and $M(F \circ G)$ be the matrices of F, G and $F \circ G$ under basis $\{e_i\}_{i=1}^n$. Then we have*

$$M(F \circ G) = M(F)M(G), \tag{3}$$

where $F \circ G$ (the composition of F and G) on the left hand side is defined as the linear map $(F \circ G)(x) = F(G(x))$ and multiplication on the the right hand side is matrix multiplication.

Proof. Exercise. □

The following problem gives the relation between the matrix of F under different basis.

Exercise 1.1. *Let $\{f_i\}_{i=1}^n$ be another basis of V, $f_i = \sum_{j=1}^n T_{ji} e_j$. Let $M(F), N(F)$ be the matrices of F under basis $\{e_i\}_{i=1}^n$, $\{f_i\}_{i=1}^n$, show that*

$$N(F) = T^{-1} M(F) T \tag{4}$$

Definition 1.1. *Let M be a matrix, the trace of M is $\operatorname{tr}(M) = \sum_{i=1}^n M_{ii}$.*

Exercise 1.2. *Let M, N be two n by n matrices, show that $\operatorname{tr}(MN) = \operatorname{tr}(NM)$.*

Exercise 1.3. *For the $N(F)$ and $M(F)$ defined above, show that $\operatorname{tr}(N(F)) = \operatorname{tr}(M(F))$.*

Definition 1.2. *Let F be a linear map on V, $M(F)$ be the matrix of F under basis $\{e_i\}_{i=1}^n$. The trace of F is defined as $\operatorname{tr} F \equiv \sum_{i=1}^n M(F)_{ii}$.*

The reader may find that we didn't specify which basis we should choose in Definition 1.2. However, by Exercise 1.3, $\operatorname{tr} F$ takes the same value under any basis, hence the definition of $\operatorname{tr} F$ has no ambiguity.

Exercise 1.4. *Let F, G be two linear maps on V, show that $\operatorname{tr}(FG) = \operatorname{tr}(GF)$.*

1.2 Direct sum

We begin by the direct sum of vector spaces.

Definition 1.3. *Let V_1 and V_2 be two vector spaces, we can define a vector space structure on the set $V_1 \times V_2$ by defining*

$$\begin{aligned} (v_1, v_2) + (w_1, w_2) &\equiv (v_1 + w_1, v_2 + w_2) \\ c(v_1, v_2) &\equiv (cv_1, cv_2) \end{aligned} \tag{5}$$

This vector space is called the direct sum of V_1 and V_2 and denoted as $V_1 \oplus V_2$.

Exercise 1.5. *Construct a isomorphism between $(V_1 \oplus V_2) \oplus V_3$ and $V_1 \oplus (V_2 \oplus V_3)$ that is independent of choice of basis. Because of the existence of this isomorphism, we can identify $(V_1 \oplus V_2) \oplus V_3$ and $V_1 \oplus (V_2 \oplus V_3)$ and simply write them as $V_1 \oplus V_2 \oplus V_3$.*

Exercise 1.6. *Let V be a vector space and V_1 and V_2 be two subspaces. Define*

$$\begin{aligned} \pi : V_1 \oplus V_2 &\to V \\ (v_1, v_2) &\mapsto v_1 + v_2 \end{aligned} \tag{6}$$

1. Show that π is injective if and only if $V_1 \cap V_2 = 0$.
2. Show that π is surjective if and only if $\forall v \in V$, $\exists v_1 \in V_1, v_2 \in V_2$ such that $v = v_1 + v_2$.
If π is bijective, we call V a direct sum of V_1 and V_2 and write $V = V_1 \oplus V_2$.

The direct sum decomposition of V are in one to one correspondence with the projection maps on V.

Definition 1.4. *A linear map $p : V \to V$ is called a projection if $p^2 = p$.*

Proposition 1.2. *If p, q are projections, then*
1. The map

$$\begin{aligned} \pi : \ker(p) \oplus \mathrm{Im}(p) &\to V \\ (x, y) &\mapsto x + y \end{aligned} \tag{7}$$

is an isomorphism.
2. If $\ker(p) = \ker(q)$ and $\mathrm{Im}(p) = \mathrm{Im}(q)$, then $p = q$.
3. If $V = V_1 \oplus V_2$, where $V_1, V_2 \subset V$, then we can find a projection p such that $\ker(p) = V_1$ and $\mathrm{Im}(p) = V_2$.
1,2,3 together tell us that the direct sum decompositions of V are in one to one correspondence with projections on V.

Proof. 1. If $v \in \ker(p) \cap \mathrm{Im}(p)$, then $v = p(w)$ for some $w \in V$ and $0 = p(v) = p^2(w) = p(w) = v$. Hence π is injective. $\forall v \in V$, we have $v = v - p(v) + p(v)$. $v - p(v) \in \ker(p), p(v) \in \mathrm{Im}(p)$, hence π is surjective.
2. $\forall v \in V$, $v = v - p(v) + p(v)$. Since $v - p(v) \in \ker(p) = \ker(q)$, we have $q(v) = q(p(v))$. Since $p(v) \in \mathrm{Im}(p) = \mathrm{Im}(q)$, we have $p(v) = q(w)$ for some $w \in V$, hence $q(v) = q(p(v)) = q(q(w)) = q(w) = p(v)$ and $p = q$.
3. If $V = V_1 \oplus V_2$, then for any $v \in V$, we can write v uniquely as $v = v_1 + v_2$, where $v_1 \in V_1$ and $v_2 \in V_2$. Define p to be $p(v) = v_2$. Then p is a projection and $\ker(p) = V_1$, $\mathrm{Im}(p) = V_2$. □

Proposition 1.2 will be used in proving the complete reducibility theorem. We now define the direct sum of linear maps, which will be useful in the character theory.

Definition 1.5. *Let $F : V_1 \to V_1$ and $G : V_2 \to V_2$ be two linear maps, the direct sum of F and G is a linear map*

$$\begin{aligned} F \oplus G : V_1 \oplus V_2 &\to V_1 \oplus V_2 \\ (v_1, v_2) &\mapsto (Fv_1, Gv_2) \end{aligned} \tag{8}$$

Exercise 1.7. *Show that $\mathrm{tr}(F \oplus G) = \mathrm{tr}(F) + \mathrm{tr}(G)$ and $\det(F \oplus G) = \det(F)\det(G)$.*

1.3 Inner product, adjoint, and unitarity

Definition 1.6. *Let V be a vector space over $\mathbb{C}$, an inner product on V is a map $\langle , \rangle : V \times V \to \mathbb{C}$ such that*
1. $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ if and only if $v = 0$.
2. $\langle v, w \rangle = \langle w, v \rangle^*$
3. $\langle v, a_1 w_1 + a_2 w_2 \rangle = a_1 \langle v, w_1 \rangle + a_2 \langle v, w_2 \rangle$

Definition 1.7. *A linear map $\alpha : V \to \mathbb{C}$ is called a functional.*

Let α, β be two functionals on V, $c \in \mathbb{C}$ be a complex number. The addition of α and β, $\alpha + \beta$, is a map $(\alpha + \beta)(v) = \alpha(v) + \beta(v)$. The scalar product $c\alpha$ is a map $(c\alpha)(v) = c\alpha(v)$.

Proposition 1.3. *The set of all functionals on V together with the addition and scalar product defined above form a vector space $V^\vee$. If V is finite dimensional, then $V^\vee$ is of the same dimension of V.*

Proof. Exercise. □

The vector space $V^\vee$ is called the dual space of V. Inner product provides a bijective map between V and $V^\vee$.

Proposition 1.4. *Let V be a finite dimensional vector space equipped with inner product $\langle , \rangle$ and $V^\vee$ be its dual. For any $v \in V$, consider the map $\phi(v) : V \to \mathbb{C}$, $\phi(v)(w) = \langle v, w \rangle, \forall w \in V$. Then $\phi(v) \in V^\vee$ and the map $\phi : V \to V^\vee$ is bijective.*

Proof. Exercise. □

Proposition 1.5. *Let V be a vector space equipped with inner product $\langle , \rangle_V$, W be a vector space equipped with inner product $\langle , \rangle_W$, for any linear map $A : V \to W$, there is a unique linear map $A^\dagger : W \to V$, called the adjoint of A, such that $\langle w, Av\rangle_W = \langle A^\dagger w, v\rangle_V, \forall v \in V, w \in W$.*

Proof. Exercise. □

Proposition 1.6. *Let V be a vector space with inner product $\langle , \rangle$, $\{v_i\}_{i=1}^n$ be an orthonormal basis of V (i.e. $\langle v_i, v_j\rangle = \delta_{ij}$). Let $A : V \to V$ be a linear map on V, whose matrix under the basis $\{v_i\}_{i=1}^n$ is $M(A)$, show that the matrix of $A^\dagger : W \to V$ under the basis $\{v_i\}_{i=1}^n$ is the conjugate transpose of $M(A)$ (denoted as $M(A)^H$).*

Proof. Exercise. □

Definition 1.8. *Let $A : V \to V$ be a linear map, A is said to be self-adjoint if $A = A^\dagger$*

In general, the eigenvalues of an operator are complex. However, the eigenvalues of a self-adjoint operator are guaranteed to be real.

Proposition 1.7. *Let A be a self-adjoint operator, all its eigenvalues are real.*

Proof. Exercise. □

Self-adjoint operator has orthogonal eigenvectors and is diagonalizable if V is finite dimensional.

Proposition 1.8. *Let A be a self-adjoint operator on V, a_1, a_2 be two distinct eigenvalues of A and v_1, v_2 be the corresponding eigenvectors. Then $\langle v_1, v_2\rangle = 0$. If V is n dimensional, then there exists orthonormal basis $\{v_1, ...v_n\}$ of V and $\{a_1, ...a_n\}$ (a_i not necessarily distinct) $Av_i = a_i v_i$.*

Proof. Exercise. □

Definition 1.9. *Let $\langle , \rangle$ be an inner product on V. A linear map U on V is said to be unitary if $\forall v, w \in V$, $\langle Uv, Uw\rangle = \langle v, w\rangle$.*

Proposition 1.9. *If U is a unitary operator, then $U^\dagger U = 1$. If $\{e_i\}_{i=1}^n$ is an orthonormal basis of V, and $M(U)$ is the matrix of U under $\{e_i\}_{i=1}^n$, then $M(U)^H M(U) = I_n$.*

Proof. Exercise. □

Definition 1.10. *An n by n matrix A is called a unitary matrix if $A^H A = AA^H = I_n$.*

Proposition 1.9 says unitary maps is represented by unitary matrices under orthonormal basis.

Proposition 1.10. *Let U be a unitary operator on V, there exists orthonormal basis $\{v_1, ...v_n\}$ of V and $\{a_1, ...a_n\}$ (a_i not necessarily distinct) with $|a_i| = 1$ and $Uv_i = a_i v_i$.*

Proof. Exercise. □

2 Basic concepts

2.1 Basic concepts of groups

Definition 2.1. *A set G together with a map (called multiplication)*

$$\begin{aligned} \pi : G \times G &\to G \\ (s,t) &\mapsto s \circ t \end{aligned} \tag{9}$$

is a group if
1. $\forall r,s,t \in G$ we have $r \circ (s \circ t) = (r \circ s) \circ t$.
2. $\exists e \in G$ such that $e \circ g = g \circ e = g, \forall g \in G$.
3. $\forall g \in G$, $\exists h \in G$ such that $g \circ h = h \circ g = e$, h is usually denoted as g^{-1}.

We usually omit $\circ$ and write $s \circ t$ as st. The definition above is abstract. In real applications, as the following examples show, groups always come from the set of operations that keep a certain object unchanged.

Example 2.1. *Let $\mathbb{R}^2$ be the Euclidean plane, $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be the distance function on $\mathbb{R}^2$, the set of distance preserving maps (a map $F : \mathbb{R}^2 \to \mathbb{R}^2$ is called distance preserving if $\forall x, y \in \mathbb{R}^2, d(F(x), F(y)) = d(x, y)$) forms a group, whose multiplication is the composition of maps. This group, denoted as $E(2)$, is called the Euclidean group. This group plays an important role in solid state physics. The elements of $E(2)$ can be classified as reflections, translations and rotations.*

Example 2.2. *Let $A = \{A_1, A_2, A_3, A_4\} \subset \mathbb{R}^2$ be the vertices of a square. Let D_4 be the set of distance preserving maps on $\mathbb{R}^2$ that keeps the set A invariant (i.e. $F(A) = A$, where $F(A) \equiv \{F(x), x \in A\}$). Then D_4 forms a group under composition of maps.*

Example 2.3. *Let $\mathbb{Z}^2$ be the square lattice on $\mathbb{R}^2$, the set of distance preserving maps on $\mathbb{R}^2$ that keeps the $\mathbb{Z}^2$ invariant forms a group under composition of maps. This group is called the space group of the square lattice.*

Definition 2.2. *A subset $H \subset G$ is called a subgroup of G if H itself forms a group under the group multiplication of G.*

Proposition 2.1. *H is a subgroup of G if and only if $\forall s, t \in H$, $s \circ t \in H$ and $s^{-1} \in H$.*

Proof. Exercise. □

Example 2.4. *The space group of the square lattice is a subgroup of $E(2)$, D_4 is also a subgroup of $E(2)$.*

Definition 2.3. *Let H be a subgroup of G. A subset T of G is said to be a coset of H if $\exists g \in G$ such that $T = gH \equiv \{gh, h \in H\}$.*

Proposition 2.2. *Let R and T be two cosets of H, then either $R = T$ or $R \cap T = \emptyset$.*

Proof. Exercise. □

Proposition 2.3. *Let R and T be two cosets of H, there is a bijective map between R and T. (Roughly speaking, this tells us that R and T have the same number of elements.)*

Proof. Exercise. □

Proposition 2.4. *Cosets of H gives a partition of G, i.e., let Q be the collection of cosets of H, then the union $\cup_{T \in Q} T$ is disjoint and $\cup_{T \in Q} T = G$.*

Proof. Use Proposition 2.2. □

Definition 2.4. *Two elements $s, t \in G$ are said to be conjugate if there exists $g \in G$ such that $s = g^{-1} t g$.*

The following examples tell that you should think of conjugate elements as elements of the same type.

Example 2.5. *Let $GL(n, \mathbb{C})$ be the set of $n \times n$ invertible matrix. Then $GL(n, \mathbb{C})$ forms a group under matrix multiplication. If two matrices M, N are conjugate, then by definition, we can find another matrix $T \in GL(n, \mathbb{C})$ such that $N = T^{-1} M T$. From Eq. (4), we know that M and N can be viewed as matrix representation of the same linear map under different basis, hence we should really think M and N are essentially the same.*

Example 2.6. *The elements of the Euclidean group $E(2)$ can be classified as reflections, translations, and rotations. Operations are conjugate if and only if they are of the same type, i.e., reflections are conjugate to each other, translations are conjugate to each other, and rotations are conjugate to each other.*

The following proposition tells the reason why conjugate elements can be viewed as the same.

Proposition 2.5. *Conjugacy is an equivalence relation, i.e.*
1. g is conjugate to g.
2. If g is conjugate to h then h is conjugate to g.
3. If g is conjugate to h, h is conjugate to k, then g is conjugate to k.

Proof. Exercise. □

Definition 2.5. *Let G and H be two groups. A map $\varphi : G \to H$ is called a homomorphism if $\varphi(st) = \varphi(s)\varphi(t), \forall s, t \in G$.*

Example 2.7. *The determinant map $\det : GL(n, \mathbb{C}) \to \mathbb{C}$ is a homomorphism.*

Definition 2.6. *Let $\varphi : G \to H$ be a homomorphism, the kernel of φ is defined as $\ker(\varphi) \equiv \{g \in G, \varphi(g) = e\}$.*

Definition 2.7. *Let N be a subgroup of G, N is called a normal subgroup of G if $\forall g \in G, n \in N$, we have $g^{-1}ng \in N$.*

Proposition 2.6. *Let $\varphi : G \to H$ be a homomorphism, $\ker(\varphi)$ is a normal subgroup of G.*

Proof. Exercise. □

Proposition 2.7. *If N is a normal subgroup of G, let G/N be the collection of cosets of N. Let $\pi : G \to G/N$ be the map that sends $g \in G$ to the coset of N that contains g. There exists a map*

$$\begin{aligned} \varphi : G/N \times G/N &\to G/N \\ (S, T) &\mapsto ST \end{aligned} \tag{10}$$

such that G/N forms a group under the map φ and π gives a homomorphism from G to G/N. The group G/N is called the quotient group of G by N.

Proof. Exercise. □

Definition 2.8. *An isomorphism is a bijective homomorphism whose inverse is also a homomorphism.*

Proposition 2.8. *Let $\varphi : G \to H$ be a homomorphism. Let $\mathrm{Im}(\varphi) = \{\varphi(g), g \in G\}$, then there exists an isomorphism $\eta : G/\ker(\varphi) \to \mathrm{Im}(\varphi)$ such that $\varphi = \eta \circ \pi$, where π is the homomorphism defined in Proposition 2.7.*

Proof. Exercise. □

2.2 Group action

Here we introduce the most important concept in group theory, the group action.

Definition 2.9. *Let S be a set, a left action of a group G on S is a map*

$$\begin{aligned} \pi : G \times S &\to S \\ (g, s) &\mapsto gs \end{aligned} \tag{11}$$

with the property
1. $es = s, \forall s \in S$
2. $g(hs) = (gh)s, \forall g, h \in G, s \in S$.

Example 2.8. *Let $E(2)$ be the Euclidean group, the map defined by*

$$\begin{aligned} \pi : E(2) \times \mathbb{R}^2 &\to \mathbb{R}^2 \\ (F, x) &\mapsto F(x) \end{aligned} \tag{12}$$

is an action of $E(2)$ on $\mathbb{R}^2$.

Example 2.9. *Let $SO(3)$ be the set of real orthogonal matrices ($g^T g = I_3$) with determinant 1. Let V be the set consisting of all functions $f : \mathbb{R}^3 \to \mathbb{C}$. $\forall g \in SO(3), f \in V$, define the function gf to be*

$$(gf)(x) = f(g^{-1}x), \forall x \in \mathbb{R}^3. \tag{13}$$

We have

$$(g(hf))(x) = (hf)(g^{-1}x) = f(h^{-1}g^{-1}x) = f((gh)^{-1}x) = ((gh)f)(x). \tag{14}$$

Hence, $g(hf) = (gh)f$. $(gf)(x) = f(g^{-1}x)$ is a action of $SO(3)$ on V.

Definition 2.10. *Let V be a vector space, $GL(V)$ be the set of invertible linear maps on V, G be a group. A linear representation of G is a homomorphism $\rho : G \to GL(V)$.*

Example 2.10. *Once we have a representation ρ of G, we can define the action of G on V by*

$$\begin{aligned} \pi : G \times V &\to V \\ (g, v) &\mapsto \rho(g)v \end{aligned} \tag{15}$$

Definition 2.11. *A vector space V together with an action of G on V is called a G-module if for any $g \in G$, the map*

$$\begin{aligned} \pi_g : V &\to V \\ v &\mapsto gv \end{aligned} \tag{16}$$

is a linear map.

G-modules and representations are in one to one correspondence.

Exercise 2.1. *Let $\mathcal{M}$ be the set of G-modules and $\mathcal{R}$ be the set of linear representations of G. Construct a bijective map between $\mathcal{M}$ and $\mathcal{R}$.*

Because of Exercise 2.1, we will identify a G-module with a representation of G and simply say V is a representation of G.

2.3 Basic concepts of representations

We begin by defining isomorphic representations.

Definition 2.12. *Let $\rho_1 : G \to GL(V)$ and $\rho_2 : G \to GL(W)$ be two representations of G, we say ρ_1 and ρ_2 are isomorphic if there is an isomorphism $F : V \to W$ so that $\forall g \in G, v \in V$, we have $F(\rho_1(g)v) = \rho_2(g)(Fv)$.*

The following exercise tells us why we should really think of ρ_1 and ρ_2 as "the same" representations.

Exercise 2.2. *If ρ_1 and ρ_2 are isomorphic, and V and W are n dimensional, we can find a base $\{v_i\}_{i=1}^n$ of V and a base $\{w_i\}_{i=1}^n$ of W so that $\forall g \in G$, the matrix of $\rho_1(g)$ under the base $\{v_i\}_{i=1}^n$ and the matrix of $\rho_2(g)$ under the base $\{w_i\}_{i=1}^n$ are the same.*

Definition 2.13. *Let V be a G-module, $W \subset V$ be a subspace, W is said to be a submodule (subrepresentation) if $\forall g \in G, w \in W$, $gw \in W$.*

Exercise 2.3. *If W is a submodule of V, then we can define a map*

$$\begin{aligned} G \times W &\to W \\ (g, w) &\mapsto gw \end{aligned} \tag{17}$$

This map makes W a G-module. The corresponding representation is denoted as $\rho|_W$.

2.4 Complete reducibility of linear representations of finite groups

The goal of linear representation theory is to classify all possible vector spaces that "has the symmetry" of G, i.e., classify all G-modules. For general groups, this is almost impossible. Here we begin to introduce the methods classifying the modules of finite groups. We begin by introducing the most important concept: the irreducible representation.

Definition 2.14. *A G-module V is said to be irreducible if $V \neq 0$ and V has no submodule except 0 and itself. Such module is also called an irreducible representation of G.*

We now show that every finite dimensional G-module V can be decomposed into direct sum of irreducible representations of G. First of all, we define the direct sum of representations.

Definition 2.15. *Let (ρ_1, W_1),..., (ρ_k, W_k) be representations of G. We define the direct sum representation of W_1,...,W_k by*

$$\begin{aligned} \rho_1 \oplus ... \oplus \rho_k : G \times (W_1 \oplus ... \oplus W_k) &\to W_1 \oplus ... \oplus W_k \\ (g, (w_1, ..., w_k)) &\mapsto (\rho_1(g)w_1, ..., \rho_k(g)w_k) \end{aligned} \tag{18}$$

Proposition 2.9. *Let (ρ, V) be a representation of G, $(\rho|_W, W)$ be a subrepresentation of G, then there exists a subrepresentation $(\rho|_{W'}, W')$ of V such that the representation $\rho|_W \oplus \rho|_{W'}$ is isomorphic to ρ.*

Proof. Let $\{v_1, ..., v_m\}$ be a base of W, we can extend it into a base $\{v_1, ..., v_m, ..., v_n\}$ of V. Let $U = \text{Span}\{v_{m+1}, ..., v_n\}$. Then it is easy to check that $V = W \oplus U$. By Proposition 1.2, we know there exists a projection P whose kernel is U, and whose image is W. Now we introduce the most important trick: **average over group**. Let n be the number of elements of G. Define

$$Q = \frac{1}{n}\sum_{g\in G}\rho(g^{-1})P\rho(g). \tag{19}$$

We claim that Q is a projection. First, $\forall v \in V$, $Qv = \frac{1}{n}\sum_{g\in G}\rho(g^{-1})P\rho(g)v$, since $P\rho(g)v \in W$, $\rho(g^{-1})P\rho(g)v \in W$ and $Qv \in W$. Hence $\operatorname{Im} Q \subset W$. $\forall w \in W$, $Qw = \frac{1}{n}\sum_{g\in G}\rho(g^{-1})P\rho(g)w = \frac{1}{n}\sum_{g\in G}\rho(g^{-1})\rho(g)w = w$. Hence $\operatorname{Im} Q = W$. $\forall v \in V$, $Qv \in W$, hence $Q^2v = Q(Qv) = Qv$ and $Q^2 = Q$. Therefore, Q is also a projection. Let $W' = \ker Q$. We claim that W' is a subrepresentation of W. $\forall w' \in W', h \in G$, we want to show $\rho(h)w' \in W'$, i.e., $Q\rho(h)w' = 0$. Indeed,

$$\begin{aligned} Q\rho(h)w' &= \frac{1}{n}\sum_{g\in G}\rho(g^{-1})P\rho(g)\rho(h)w' \\ &= \frac{1}{n}\sum_{g\in G}\rho(h)\rho(h^{-1})\rho(g^{-1})P\rho(gh)w' \\ &= \frac{1}{n}\sum_{g\in G}\rho(h)\rho(h^{-1}g^{-1})P\rho(gh)w' \\ &= \frac{1}{n}\sum_{g\in G}\rho(h)\rho((gh)^{-1})P\rho(gh)w', \end{aligned} \tag{20}$$

where we used $h^{-1}g^{-1} = (gh)^{-1}$ in the last step. Now, the **key observation** is that if g runs over G, gh also runs over G. We actually have

$$\begin{aligned} Q\rho(h)w' &= \frac{1}{n}\sum_{g\in G}\rho(h)\rho((gh)^{-1})P\rho(gh)w' \\ &= \rho(h)\frac{1}{n}\sum_{g\in G}\rho((gh)^{-1})P\rho(gh)w' \\ &= \rho(h)\frac{1}{n}\sum_{t\in G}\rho(t^{-1})P\rho(t)w' \\ &= \rho(h)Qw' \\ &= 0. \end{aligned} \tag{21}$$

The last step is because $w' \in W' = \ker Q$. Hence W' is also a subrepresentation of G. Now consider the map

$$\begin{aligned} \pi : W \oplus W' &\to V \\ (w, w') &\mapsto w + w' \end{aligned} \tag{22}$$

From Proposition 1.2 we know π is an isomorphism between vector spaces. $\forall w \in W, w' \in W', g \in G$, we have

$$\begin{aligned} \pi((\rho|_W \oplus \rho|_{W'})(g)((w, w'))) &= \pi(\rho(g)w, \rho(g)w') \\ &= \rho(g)w + \rho(g)w' \\ &= \rho(g)(w + w') \\ &= \rho(g)\pi((w, w')). \end{aligned} \tag{23}$$

Hence, $\pi(\rho|_W \oplus \rho|_{W'})(g) = \rho(g)\pi$, and $\rho|_W \oplus \rho|_{W'}$ is isomorphic to ρ. ☐

The following exercise is an application of the trick of average over group.

Exercise 2.4. *Let (ρ, V) be a representation of G, the goal of this exercise is to show that we can always find a base $\{v_i\}_{i=1}^n$ of V such that $\forall g \in G$, the matrix of $\rho(g)$ under $\{v_i\}_{i=1}^n$ is unitary.*
Let $\{w_i\}_{i=1}^n$ be an arbitrary base of V, define a inner product $\langle , \rangle$ on V by

$$\langle \sum_{i=1} a_i w_i, \sum_{i=1} b_i w_i \rangle \equiv \sum_{i=1}^{n} a_i^* b_i. \tag{24}$$

Let $|G|$ be the number of elements in G and define

$$\langle u, v\rangle_1 \equiv \frac{1}{|G|}\sum_{s\in G}\langle \rho(s)u, \rho(s)v\rangle. \tag{25}$$

1. Show that $\langle , \rangle_1$ is also an inner product.
2. Show that $\langle \rho(h)u, \rho(h)v\rangle_1 = \langle u, v\rangle_1, \forall h \in G, u, v \in V$.
3. Let $\{v_i\}_{i=1}^n$ be an orthonormal base of V under the inner product $\langle , \rangle_1$, show that $\forall g \in G$, the matrix of $\rho(g)$ under $\{v_i\}_{i=1}^n$ is unitary.

Theorem 2.1. *Let V be a finite dimensional representation of a finite group G. Then V is isomorphic to a direct sum irreducible representations of G.*

Proof. We prove by doing induction on the dimension of V. $\dim V = 0$ is obvious as it is the direct sum of empty family of irreducible representations. Suppose the theorem is true for $\dim V \leq n$. Then for $\dim V = n + 1$, if V is already irreducible, we are done. If V is not irreducible, by definition, we can always find a proper subrepresentation $W \subset V$. Then use Proposition 2.9, we can find a subrepresentation $W' \subset V$ so that the representation $W \oplus W'$ is isomorphic to V (we will simply write it as $V = W \oplus W'$). Since $\dim W \leq n, \dim W' \leq n$, we can apply the induction hypothesis to W and W' and prove that V is isomorphic to a direct sum of irreducible representations. □

Because of this theorem, we only need to focus on the irreducible representations, and we say linear representation of finite groups is completely reducible.

2.5 The canonical representation

In this section we introduce the most important example in the theory of linear representation of finite group: the canonical representation.

Example 2.11. *Let V be the set consisting of all functions $f : G \to \mathbb{C}$. $\forall f, h \in V$, define $(f + h)(g) = f(g) + h(g), \forall g \in G$ and $(cf)(g) = cf(g), \forall g \in G, c \in \mathbb{C}$. These operations make V a vector space. Now, $\forall g \in G, f \in V$, define gf to be the function*

$$(gf)(h) = f(g^{-1}h), \forall h \in G. \tag{26}$$

Using the same argument in Example 2.9, we know that this makes V a G-module. This representation is called the canonical representation of G.

Exercise 2.5. *Show that V is $|G|$ dimensional, where $|G|$ is the number of elements in G by showing that the functions $\{\delta_s\}_{s\in G}$, where $\delta_s(g) = 1$ if $g = s$ and $\delta_s(g) = 0$ if $g \neq s$ is a base of V.*

Exercise 2.6. *Show that $g\delta_s = \delta_{gs}$.*

Example 2.11 is of fundamental importance in the representation theory a finite groups. As will be shown in the next chapter, the canonical representation contains all irreducible representations of a finite group G.

3 Character theory

3.1 The character of a representation

Definition 3.1. *Let $\rho : G \to GL(V)$ be a representation, the character of ρ is a map*

$$\begin{aligned} \chi_\rho : & G \to \mathbb{C} \\ & g \mapsto \mathrm{tr}(\rho(s)) \end{aligned} \tag{27}$$

Proposition 3.1. *Let $\rho : G \to GL(V)$ be an n dimensional representation ($\dim V = n$). Then we have,*
1. $\chi_\rho(e) = n$, where e is the identity element in G.
2. $\chi_\rho(s^{-1}) = \chi_\rho(s)^*$.
3. $\chi_\rho(t^{-1}st) = \chi_\rho(s)$.

Proof. 1. $\rho(e)\rho(e) = \rho(ee) = \rho(e)$, since $\rho(e) \in GL(V)$, we can multiply $\rho(e)^{-1}$ on both side and obtain $\rho(e) = \mathrm{Id}$, where Id is the identity map on V. The matrix of Id under any base is the unit n by n matrix I_n. Hence $\chi_\rho(e) = \mathrm{tr}(\mathrm{Id}) = \sum_{i=1}^n (I_n)_{ii} = \sum_{i=1}^n 1 = n$.
2. From Exercise 2.4, we know that we can find a base $\{v_i\}_{i=1}^n$ of V such that the matrix of $\rho(s)$, denoted by $M(\rho(s))$ is a unitary matrix. Using Proposition 1.1, we know that

$$M(\rho(s^{-1}))M(\rho(s)) = M(\rho(s^{-1})\rho(s)) = M(\rho(s^{-1}s)) = M(\rho(e)) = I_n, \tag{28}$$

Hence, $M(\rho(s^{-1})) = M(\rho(s))^{-1} = M(\rho(s))^H$, where the last step is from Proposition 1.9. Now we can compute

$$\chi_\rho(s^{-1}) = \mathrm{tr}(\rho(s^{-1})) = \sum_{i=1}^n M(\rho(s^{-1}))_{ii} = \sum_{i=1}^n (M(\rho(s))^H)_{ii} = \sum_{i=1}^n (M(\rho(s))_{ii})^* = \chi_\rho(s)^*. \tag{29}$$

3. Using $\mathrm{tr}(FG) = \mathrm{tr}(GF)$, we see that

$$\chi_\rho(t^{-1}st) = \mathrm{tr}(\rho(t^{-1})\rho(s)\rho(t)) = \mathrm{tr}(\rho(s)\rho(t)\rho(t^{-1})) = \mathrm{tr}(\rho(s)) = \chi_\rho(s). \tag{30}$$

□

Proposition 3.2. *Let (ρ, V) be a representation of G, and (ρ_1, V_1) and (ρ_2, V_2) be two subrepresentations of V and $V = V_1 \oplus V_2$. The characters of ρ, ρ_1, ρ_2 are χ, χ_2, χ_2. We have $\chi = \chi_1 + \chi_2$*

Proof. Let $\{e_i\}_{i=1}^m$ be a base of V_1 and $\{f_i\}_{i=1}^n$ be a base of V_2. Then $\{e_1, ..., e_m, f_1, ..., f_n\}$ form a base of V. $\forall s \in G$, write down the matrix of $\rho(s)$ with respect to $\{e_1, ..., e_m, f_1, ..., f_n\}$ and the result follows. □

3.2 Schur's lemma and its basic applications

In this section, we first introduce the Schur's lemma, which is of **fundamental importance** in group representation theory. We then talk about its basic applications.

Proposition 3.3. *Let $\rho_1 : G \to GL(V_1)$, $\rho_2 : G \to GL(V_2)$ be two irreducible representations. Let $F : V_1 \to V_2$ be a linear map such that $\forall s \in G$, $F\rho_1(s) = \rho_2(s)F$.*
1. If ρ_1 and ρ_2 are not isomorphic, then $F = 0$.
2. If $V_1 = V_2 = V$, $\rho_1 = \rho_2 = \rho$, then $F = \lambda\,\mathrm{Id}$.

Proof. 1. We begin by showing that $\ker(F)$ is a subrepresentation of V_1 and $\mathrm{Im}(F)$ is a subrepresentation of V_2. $\forall v_1 \in \ker(F), s \in G$, $F\rho_1(s)v_1 = \rho_2(s)Fv_1 = 0$, hence $\rho_1(s)v_1 \in \ker(F)$ and $\ker(F)$ is a subrepresentation of V_1. Since V_1 is irreducible, $\ker(F) = 0$ or $\ker(F) = V_1$. $\forall Fv_1 \in \mathrm{Im}(F), s \in G$, $\rho_2(s)Fv_1 = F\rho_1(s)v_1 \in \mathrm{Im}(F)$, hence $\mathrm{Im}(F)$ is a subrepresentation of V_2. Since V_2 is irreducible, $\mathrm{Im}(F) = 0$ or $\mathrm{Im}(F) = V_2$. If $\ker(F) = 0$, then we must have $\mathrm{Im}(F) = V_2$, this implies F is an isomorphism between ρ_1 and ρ_2, contradicts with the assumption that ρ_1 and ρ_2 are not isomorphic. So we must have $\ker(F) = V_1$ and hence $F = 0$.
2. Let λ be an eigenvalue of F (we know it must exist as we are working on vector spaces over $\mathbb{C}$), consider $G = F - \lambda\,\mathrm{Id}$. Then $G\rho(s) = \rho(s)G$. Repeating the argument used in the first part we know either $\ker(G) = 0$ or $\ker(G) = V$. However, G by construction has zero eigenvalue and $\ker(G) \neq 0$. Therefore, $\ker(G) = V$, $G = 0$ and $F = \lambda\,\mathrm{Id}$. □

Corollary 3.1. *Let $\rho_1 : G \to V_1$, $\rho_2 : G \to V_2$ be two irreducible representations, H be a linear map from V_1 to V_2, and $|G|$ be the number of elements in G. Let*

$$J = \frac{1}{|G|} \sum_{g \in G} \rho_2(s^{-1}) H \rho_1(s). \tag{31}$$

1. If ρ_1 and ρ_2 are not isomorphic, $J = 0$.
2. If $V_1 = V_2 = V$, $\dim V = n$, and $\rho_1 = \rho_2 = \rho$, then

$$J = \frac{1}{n}\operatorname{tr}(H)\operatorname{Id}. \tag{32}$$

Proof. 1. Using the same trick as that in Proposition 2.9, we can show that $\rho_2(s)J = J\rho_1(s), \forall s \in G$. Then use Proposition 3.3 we know that $J = 0$.
2. Use Proposition 3.3 we know that $J = \lambda \operatorname{Id}$, take trace on both side we obtain $\operatorname{tr}(J) = \lambda n$. From the definition of J we know $\operatorname{tr}(J) = \operatorname{tr}(H)$ and the result follows. □

Corollary 3.2. *Under the same set up of Corollary 3.1,*
1. If ρ_1 and ρ_2 are not isomorphic, pick arbitrary basis $\{u_i\}_{i=1}^m$ and $\{v_i\}_{i=1}^n$ of V_1 and V_2, we can write down the matrix of $\rho_1(s)$ as $M_1(s)$ and the matrix of $\rho_2(s)$ as $M_2(s)$, we have

$$\frac{1}{|G|}\sum_{s\in G} M_2(s^{-1})_{i_2 j_2} M_1(s)_{j_1 i_1} = 0, \forall i_1, j_1, i_2, j_2. \tag{33}$$

2. If $V_1 = V_2 = V$, $\dim V = n$, and $\rho_1 = \rho_2 = \rho$, pick arbitrary basis $\{v_i\}_{i=1}^n$ of V, we can write down the matrix of $\rho(s)$ as $M(s)$. We have

$$\frac{1}{|G|}\sum_{s\in G} M(s^{-1})_{i_2 j_2} M(s)_{j_1 i_1} = \frac{1}{n}\delta_{i_2 i_1}\delta_{j_2 j_1}, \forall i_1, j_1, i_2, j_2. \tag{34}$$

Proof. 1. Let the matrix of H, J under basis $\{u_i\}_{i=1}^m$ and $\{v_i\}_{i=1}^n$ be $M(H)$ and $M(J)$. Then we have

$$0 = M(J) = \frac{1}{|G|}\sum_{g\in G} M_1(g^{-1})M(H)M_2(g). \tag{35}$$

Take the $i_2 i_1$ component of $M(J)$ we get

$$0 = \frac{1}{|G|}\sum_{j_2=1}^{n}\sum_{j_1=1}^{m}\sum_{g\in G} M_2(g^{-1})_{i_2 j_2} M(H)_{j_2 j_1} M_1(g)_{j_1 i_1} \tag{36}$$

Since $M(H)_{j_2 j_1}$ is arbitrary, the coefficient of $M(H)_{j_2 j_1}$ must be zero and we obtain the desired result.
2. Let the matrix of H, J under base $\{v_i\}_{i=1}^n$ be $M(H)$ and $M(J)$. Take the $i_2 i_1$ component of $M(J)$, we have

$$\frac{1}{n}\sum_{k=1}^{n} M(H)_{kk}\delta_{i_2 i_1} = \frac{1}{|G|}\sum_{j_2=1}^{n}\sum_{j_1=1}^{n}\sum_{g\in G} M(g^{-1})_{i_2 j_2} M(H)_{j_2 j_1} M(g)_{j_1 i_1} \tag{37}$$

The left hand side can be written as

$$\frac{1}{n}\sum_{k=1}^{n} M(H)_{kk}\delta_{i_2 i_1} = \frac{1}{n}\sum_{j_2=1}^{n}\sum_{j_1=1}^{n} M(H)_{j_2 j_1}\delta_{j_1 j_2}\delta_{i_2 j_1} \tag{38}$$

Since $M(H)_{j_2 j_1}$ is arbitrary, the coefficient of $M(H)_{j_2 j_1}$ on the left hand side and right hand side must be the same, and we obtain the desired formula. □

3.3 The space of class functions

Definition 3.2. *A function $f : G \to \mathbb{C}$ is a class function if $f(t^{-1}gt) = f(g), \forall t, g \in G$, i.e., f take the same value on elements in the same conjugate class.*

Characters are class functions. Let H be the set of class functions. H forms a vector space under the usual addition of functions and scalar multiplication. Let f_s be the function that equals 1 on conjugate class s and equals 0 on other conjugate classes. $\{f_s\}$ forms a base of H and the dimension of H is the number of conjugate classes in G. We can define an inner product in the following way. $\forall \psi, \chi \in H$ define their inner product by

$$\langle \psi, \chi \rangle \equiv \frac{1}{|G|}\sum_{g\in G} \psi(g)^* \chi(g). \tag{39}$$

In the following section, we will show that the set of characters of irreducible representations form an orthonormal base of H.

3.4 Orthogonality relations for characters

We first show that characters of irreducible representations are orthogonal to each other under the inner product defined in Eq. (39).

Proposition 3.4. *If χ_1, χ_2 are the characters of two non-isomorphic irreducible representation (of degree n and m respectively) of G, then*

$$\begin{aligned} \langle \chi_1, \chi_1 \rangle &= 1 \\ \langle \chi_1, \chi_2 \rangle &= 0 \end{aligned} \tag{40}$$

Proof. Let $j_1 = i_1$, $j_2 = i_2$ in Eq. (34) and sum over i_1 and i_2 we have

$$\sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \frac{1}{|G|} \sum_{s \in G} M(s^{-1})_{i_2 i_2} M(s)_{i_1 i_1} = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \frac{1}{n} \delta_{i_2 i_1} \delta_{i_2 i_1} = 1 \tag{41}$$

The left hand side is

$$\begin{aligned} \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \frac{1}{|G|} \sum_{s \in G} M(s^{-1})_{i_2 i_2} M(s)_{i_1 i_1} &= \frac{1}{|G|} \sum_{s \in G} \sum_{i_2=1}^{n} M(s^{-1})_{i_2 i_2} \sum_{i_1=1}^{n} M(s)_{i_1 i_1} \\ &= \frac{1}{|G|} \sum_{g \in G} \chi_1(g^{-1}) \chi_1(g) \\ &= \frac{1}{|G|} \sum_{g \in G} \chi_1(g)^* \chi_1(g) \\ &= \langle \chi_1 \chi_1 \rangle. \end{aligned} \tag{42}$$

The right hand side is 1. Hence $\langle \chi_1, \chi_1 \rangle = 1$. Let $j_1 = i_1$, $j_2 = i_2$ in Eq. (33) and sum over i_1 and i_2 we have

$$\sum_{i_1=1}^{n} \sum_{i_2=1}^{m} \frac{1}{|G|} \sum_{s \in G} M_2(s^{-1})_{i_2 j_2} M_1(s)_{j_1 i_1} = 0. \tag{43}$$

Using the same argument, the left hand side is $\langle \chi_1, \chi_2 \rangle$, and $\langle \chi_1, \chi_2 \rangle = 0$. □

An immediate consequence of Proposition 3.4 is that different irreducible representations must have different characters. Hence a finite group G can only have finite number of irreducible representations (as the characters of them form an orthogonal system of a finite dimensional vector space H). Now we give some applications of Proposition 3.4. First, we study the irreducible decomposition. Theorem 2.1 shows that any representation can be decomposed into direct sum of irreducible representations. This decomposition, however, is in general not unique. Here we show that the number of appearence of an irreducible representation in an irreducible decomposition is independent of the irreducible decomposition you choose. Let V be a representation of G and χ be its character. Let the irreducible representations of G be $(\rho_1, V_1), ..., (\rho_h, V_h)$ and $\chi_1, ..., \chi_h$ be the corresponding characters. Let $V = W_1 \oplus ... \oplus W_k$ be an arbitrary irreducible decomposition of V (note this decomposition is in general not unique).

Theorem 3.1. *Let m_k be the number of irreducible representation isomorphic to V_k in the irreducible decomposition above, we have $m_k = \langle \chi_k, \chi \rangle$, and m_k is independent of the decomposition.*

Proof. This is an immediate consequence of Proposition 3.4 and Proposition 3.2. □

Because of Theorem 3.1, we usually write the irreducible decomposition as $V = m_1 V_1 \oplus ... \oplus m_h V_h$. Now we give a criterion for determining whether two representations are isomorphic.

Theorem 3.2. *Two representations are isomorphic if and only if they have the same characters.*

Proof. The only if part is easy. For the if part, note that if they have the same characters, by Theorem 3.1, they have the same number of copies of irreducible representation V_k for each $1 \leq k \leq h$ and hence they are isomorphic. □

Finally, we give a criterion for irreducibility.

Theorem 3.3. *Let (ρ, V) be a representation of G and χ be its character, then ρ is irreducible if and only if $\langle \chi, \chi \rangle = 1$.*

Proof. The only if part if easy. For the if part, let $V = m_1 V_1 \oplus ... \oplus m_h V_h$, then $1 = \langle \chi, \chi \rangle = \sum_{i=1}^{h} m_i^2$. Hence only one m_i is 1 and all other m_j are zero, and the representation is irreducible. □

Proposition 3.4 says the irreducible characters form an orthogonal system of H. Now we further show that they form an orthonormal basis of H. We begin by the trick of averaging over group.

Proposition 3.5. *Let $f \in H$ be a class function, $\rho : G \to V$ be an irreducible representation of degree n (meaning $\dim V = n$ with character χ. Let ρ_f be the linear map on V defined by*

$$\rho_f = \frac{1}{|G|} \sum_{g \in G} f(g)\rho(g). \tag{44}$$

Then $\rho_f = \lambda \, \mathrm{Id}$, with

$$\lambda = \frac{1}{n} \langle \chi^*, f \rangle. \tag{45}$$

Proof. $\forall s \in G$, we have

$$\begin{aligned} \rho(s^{-1})\rho_f\rho(s) &= \frac{1}{|G|} \sum_{g \in G} f(g)\rho(s^{-1})\rho(g)\rho(s) \\ &= \frac{1}{|G|} \sum_{g \in G} f(g)\rho(s^{-1}gs) \\ &= \frac{1}{|G|} \sum_{g \in G} f(s^{-1}gs)\rho(s^{-1}gs) \\ &= \frac{1}{|G|} \sum_{t \in G} f(t)\rho(t) \\ &= \rho_f. \end{aligned} \tag{46}$$

Third equation uses $f(s^{-1}gs) = f(g)$. The fourth equation uses the fact that for any fixed $s \in G$, $s^{-1}gs$ runs over G when g runs over G. Hence we have $\rho_f\rho(s) = \rho(s)\rho_f$. We may then use Proposition 3.3 to conclude that $\rho_f = \lambda \, \mathrm{Id}$. Take trace on both side we have

$$n\lambda = \frac{1}{|G|} \sum_{g \in G} f(g)\chi(g) = \langle \chi^*, f \rangle, \tag{47}$$

and we have $\lambda = \langle \chi^*, f \rangle / n$. □

With Proposition 3.5, we are ready to prove that irreducible characters forms orthonormal base of H.

Theorem 3.4. *Irreducible characters of G form an orthonormal base of H.*

Proof. Let $\chi_1, ..., \chi_h$ be all irreducible characters of G. First we prove that if a class function f is orthogonal to all χ_k^* (where $\chi_k^*(s) \equiv \chi_k(s)^*$), then $f = 0$. For this, let ρ be the canonical representation of G (Example 2.11) and χ be its character. By Theorem 2.1, χ is a linear combination of $\chi_1, ..., \chi_h$. Hence if f is orthogonal to all χ_k^*, $\langle \chi^*, f \rangle = 0$ and $\rho_f = 0$. Using Exercise 2.6, acting ρ_f on δ_e, where e is the unit element of G, we have

$$\rho_f \delta_e = \frac{1}{|G|} \sum_{g \in G} f(g)\rho(g)\delta_e = \frac{1}{|G|} \sum_{g \in G} f(g)\delta_g = 0. \tag{48}$$

Exercise 2.5 tells us that δ_g are linearly independent. Hence $f(g) = 0, \forall g \in G$ and $f = 0$.
Now we prove $\chi_1, ..., \chi_h$ form a orthonormal base of H. It's enough to prove that $\forall f \in H$, f can be written as a linear combination of $\chi_1, ..., \chi_h$. Let $q = f - \sum_{k=1}^{h} \langle \chi_k, f \rangle \chi_k$. Then by the definition of $\langle \chi_k, f \rangle$, we have

$$\langle \chi_k, f \rangle^* = \langle \chi_k^*, f^* \rangle. \tag{49}$$

Hence $q^* = f^* - \sum_{k=1}^{h} \langle \chi_k^*, f^* \rangle \chi_k^*$. Using $\langle \chi_k^*, \chi_j^* \rangle = \langle \chi_k, \chi_j \rangle^* = \delta_{kj}$, we see $\langle \chi_r^*, q^* \rangle = 0, \forall 1 \le r \le h$, and $q^* = 0$. Hence $q = 0$ and

$$f = \sum_{k=1}^{h} \langle \chi_k, f \rangle \chi_k, \tag{50}$$

and $\chi_1, ..., \chi_h$ form a orthonormal base of H. □

Theorem 3.5. *The number of irreducible representations of G equals the number of conjugate classes of G.*

Proof. The number of irreducible representations of G equals the dimension of H, which equal the number of conjugate classes of G. □

We now provide two more relations between the irreducible characters.

Proposition 3.6. *Let $s, t \in G$ be in different conjugate class, and $c(s)$ be the number of elements in the conjugate class of s, we have*

$$\begin{aligned} \sum_{k=1}^{h} \chi_k(s)^* \chi_k(s) &= \frac{|G|}{c(s)} \\ \sum_{k=1}^{h} \chi_k(s)^* \chi_k(t) &= 0 \end{aligned} \tag{51}$$

Proof. Let f_s be the class function that equals 1 on the conjugate class of s and equals 0 on the other elements. Using Eq. (50), we have

$$f_s = \sum_{k=1}^{h} \langle \chi_k, f_s \rangle \chi_k. \tag{52}$$

By definition,

$$\langle \chi_k, f_s \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_k(g)^* f_s(g) = \frac{c(s)}{|G|} \chi_k^*(s). \tag{53}$$

Hence we have

$$f_s = \frac{c(s)}{|G|} \sum_{k=1}^{h} \chi_k(s)^* \chi_k. \tag{54}$$

Using $f_s(s) = 1, f_s(t) = 0$ we obtain the desired results. □

3.5 Decomposition of the canonical representation

Proposition 3.7. *Let (ρ, V) be the canonical representation of G (Example 2.11), $(\rho_1, V_1), ..., (\rho_h, V_h)$ be the irreducible representations of G of degree $n_1, ..., n_h$, and $\chi_1, ..., \chi_h$ be the corresponding characters. Then we have*

$$V = n_1 V_1 \oplus ... \oplus n_h V_h \tag{55}$$

Proof. Let χ be the character of ρ. It's easy to compute

$$\begin{aligned} \chi(e) &= |G| \\ \chi(s) &= 0, \forall s \neq e \end{aligned} \tag{56}$$

Let $V = m_1 V_1 \oplus ... \oplus m_h V_h$, then from Theorem 3.1, we know

$$m_k = \langle \chi_k, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_k(g)^* \chi(g) = \frac{1}{|G|} \chi_k(e)^* \chi(e) = \chi_k(e)^* = n_k. \tag{57}$$

□

Proposition 3.7 tells us that the canonical representation contains all the irreducible representations, it also provides us with two more identities for the irreducible characters.

Corollary 3.3. *We have the following identities*

$$\begin{aligned} \sum_{k=1}^{h} n_k^2 &= |G| \\ \sum_{k=1}^{h} n_k \chi_k(s) &= 0, \forall s \neq e. \end{aligned} \tag{58}$$

Proof. By Eq. (50), we have $\chi = \sum_{k=1}^{h} n_k \chi_k$. $\chi(e) = |G|$ gives us the first identity and $\chi(s) = 0$ gives us the second identity. □

Note that Corollary 3.3 can also be obtained from Eq. (51).

3.6 Example: the character table for D_4

In this section we work the character table of D_4 (Example 2.2). D_4 contains 8 elements: the identity element Id, rotation by $\frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and four reflections. Let $Ro(\theta)$ be the rotation (counterclockwise) by θ and $Re(\theta)$ be the reflection along the line whose angle (counterclockwise) between the x-axis is θ. We have

$$\begin{aligned} Re(\theta)^{-1}Ro(\alpha)Re(\theta) &= Ro(-\alpha) \\ Ro(\theta)^{-1}Re(\alpha)Ro(\theta) &= Re(\alpha+\theta) \end{aligned} \tag{59}$$

Hence D_4 is divided into 5 conjugate classes

$$D_4 = \{\{\mathrm{Id}\}, \{Ro(\pi)\}, \{Ro(\frac{\pi}{2}), Ro(\frac{3\pi}{2})\}, \{Re(0), Re(\frac{\pi}{2})\}, \{Re(\frac{\pi}{4}), Re(\frac{3\pi}{4})\}\}. \tag{60}$$

Let the classes be c_1, c_2, c_3, c_4, c_5 respectively. From Theorem 3.5, D_4 has 5 irreducible representations $\chi_1, ..., \chi_5$. Using Corollary 3.3, we know one of the irreducible representation has degree 2 and all the other 4 has degree 1. Suppose χ_5 has degree 2. First of all, we have two degree 1 representations: ρ_1 that maps every group element to 1 and ρ_2 that maps $g \in D_4$ to $\det(g)$. Since by definition, representations of degree 1 are irreducible, these two irreducible representations gives us irreducible characters χ_1 and χ_2. Second, notice that D_4 is already a group consisting of linear maps on $\mathbb{R}^2$, i.e., $D_4 \subset GL(\mathbb{R}^2)$, we have a tautological representation $\rho_5; D_4 \to GL(\mathbb{R}^2)$, where ρ_5 is just the identity map. Using Theorem 3.3, we see this is an irreducible representation and we have the last row in Table 1. Third, since we know χ_3 and χ_4 are of degree 1, we can know the first column. Now we have 8 unknowns. Orthogonality relations (Proposition 3.4) provides 6 linear constraints for χ_3 and χ_4 and Eq. (51) provides another 4 linear constraints. Solving these linear equations we obtain Table 1.

	c_1	c_2	c_3	c_4	c_5
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

Table 1: The character table of D_4

4 An application of character theory: block diagonalization

4.1 Canonical decomposition of a representation

In general, there are multiple ways of decomposing a given representation into the direct sum of irreducible representations. Here we introduce a canonical way of decomposing representations called the canonical decomposition. As will be seen later, canonical decomposition blockdiagonalizes symmetric Hamiltonian.

Proposition 4.1. *Let (ρ, V) be a linear representation of G, $\{(\rho_j, V_j)\}_{j=1}^h$ be the irreducible representations of G, each with degree n_j and character χ_j, $V = \oplus_{j=1}^h m_j V_j$ be an arbitrary irreducible decomposition of V, and $U_j \equiv m_j V_j$. Then the operators*

$$p_j \equiv \frac{n_j}{|G|} \sum_{g \in G} \chi_j(g)^* \rho(g) \tag{61}$$

are projection operators, and $p_j(V) = U_j$. In other words, the decomposition $V = \oplus_{j=1}^h U_j$ is independent of the choice of irreducible representations. This decomposition is called the canonical decomposition.

Proof. Let $W \subset V$ be an irreducible subrepresentation of degree n and character χ, then by definition, p_j maps W to W and we denote the restriction of p_j to W by p_j^W. We have

$$\begin{aligned} \rho|_W(s^{-1}) p_j^W \rho|_W(s) &= \frac{n_j}{|G|} \sum_{g \in G} \chi_j(g)^* \rho|_W(s^{-1}) \rho|_W(g) \rho|_W(s) \\ &= \frac{n_j}{|G|} \sum_{g \in G} \chi_j(g)^* \rho|_W(s^{-1} g s) \\ &= \frac{n_j}{|G|} \sum_{g \in G} \chi_j(s^{-1} g s)^* \rho|_W(s^{-1} g s) \\ &= \frac{n_j}{|G|} \sum_{t \in G} \chi_j(t)^* \rho|_W(t) \\ &= p_j^W. \end{aligned} \tag{62}$$

By Schur's lemma, $p_j^W = \lambda \, \mathrm{Id}_W$, take trace on both side we have

$$\begin{aligned} \operatorname{tr} p_j^W &= \frac{n_j}{|G|} \sum_{g \in G} \chi_j(g)^* \operatorname{tr} \rho|_W(g) \\ &= \frac{n_j}{|G|} \sum_{g \in G} \chi_j(g)^* \chi(g) \\ &= n_j \langle \chi_j, \chi \rangle \\ &= \lambda n_j \end{aligned} \tag{63}$$

Hence if W is isomorphic to V_j, $p_j^W = \mathrm{Id}_W$. Otherwise, $p_j^W = 0$. Since U_i is the direct sum of irreducible representations isomorphic to V_i, we have $p_j^{U_i} = \delta_{ij} \, \mathrm{Id}_{U_i}$. Hence p_j is a projection operator for all j and $p_j(V) = U_j$. ☐

4.2 Block diagonalization of symmetric Hamiltonian

Here we introduce a general method of block diagonalizing Hamiltonians using the canonical decomposition of the Hilbert space under the symmetry group of the Hamiltonian. The general set up in physics is the following. Let $\mathbb{H}$ be the Hilbert space of certain physical system, H be the Hamiltonian of the system. It is often the case that we can find some symmetries of the physical systems before doing actual calculations. These symmetries corresponds to invertible linear operators commuting with the Hamiltonian. Once we found several linear operators commuting with H, we can use them to generate a group G, and the Hilbert space $\mathbb{H}$ carries a representation of G. The following theorem tells us how to block diagonalize H with G.

Theorem 4.1. *Let (ρ, V) be a representation of G, $V = \oplus_{i=1}^h U_i$ be the canonical decomposition of V, $H : V \to V$ be a linear map that has G as a symmetry group, i.e. $[H, \rho(g)] \equiv H\rho(h) - \rho(g)H = 0, \forall g \in G$. Then U_j is an invariant subspace of H.*

Proof. $\forall w_j \in U_j$, we have

$$p_j H w_j = \frac{n_j}{|G|} \sum_{g \in G} \chi_j(g)^* \rho(g) H w_j = H \frac{n_j}{|G|} \sum_{g \in G} \chi_j(g)^* \rho(g) w_j = H p_j w_j = H w_j, \tag{64}$$

hence $H w_j \in U_j$ and U_j is an invariant subspace of H. ☐

5 The group algebra

5.1 The group algebra $\mathbb{C}[G]$

The group algebra $\mathbb{C}[G]$ is a vector space whose basis is the set G, together with a multiplication induced by the group multiplication. It is useful because it turns the study of representations of G into the study of modules over one algebra.

Definition 5.1. *An algebra over $\mathbb{C}$ is a vector space A over $\mathbb{C}$ equipped with a multiplication map*

$$A \times A \to A, \qquad (a, b) \mapsto ab,$$

such that

1. *the multiplication is bilinear;*
2. *$(ab)c = a(bc)$ for all $a, b, c \in A$;*
3. *there exists an element $1_A \in A$ such that $1_A a = a 1_A = a$ for all $a \in A$.*

The element 1_A is called the unit of A.

Definition 5.2. *Let G be a finite group. The group algebra $\mathbb{C}[G]$ is the vector space consisting of formal sums*

$$\sum_{g \in G} c_g g, \qquad c_g \in \mathbb{C}.$$

Addition and scalar multiplication are defined coefficient-wise. Multiplication is defined by extending the group multiplication linearly:

$$\left(\sum_{g \in G} c_g g \right) \left(\sum_{s \in G} d_s s \right) = \sum_{g, s \in G} c_g d_s (gs).$$

The unit of $\mathbb{C}[G]$ is the identity element $e \in G$, viewed as an element of the basis of $\mathbb{C}[G]$.

Proposition 5.1. *$\mathbb{C}[G]$ is an algebra over $\mathbb{C}$.*

Proof. Bilinearity follows from the definition. Associativity follows from associativity of the group multiplication:

$$((g_1 g_2) g_3) = g_1 (g_2 g_3), \qquad g_1, g_2, g_3 \in G,$$

and then by linearity for arbitrary elements of $\mathbb{C}[G]$. The unit is the identity element $e \in G$. □

Definition 5.3. *Let A be an algebra over $\mathbb{C}$. A left A-module is a vector space V together with a bilinear map*

$$A \times V \to V, \qquad (a, v) \mapsto av,$$

such that

$$(ab)v = a(bv), \qquad 1_A v = v,$$

for all $a, b \in A$ and $v \in V$.

Proposition 5.2. *Giving a representation of G on V is equivalent to giving a left $\mathbb{C}[G]$-module structure on V.*

Proof. Suppose first that V is a representation of G. Thus each $g \in G$ acts linearly on V. Define

$$\left(\sum_{g \in G} c_g g \right) v = \sum_{g \in G} c_g (gv).$$

This gives a left $\mathbb{C}[G]$-module structure on V because

$$\left(\sum_g c_g g \right) \left(\left(\sum_s d_s s \right) v \right) = \sum_{g,s} c_g d_s \, g(sv) = \sum_{g,s} c_g d_s \, (gs) v.$$

Conversely, if V is a left $\mathbb{C}[G]$-module, then each basis element $g \in G \subset \mathbb{C}[G]$ acts linearly on V. Since e acts as the identity and $(gs)v = g(sv)$, this gives an action of G on V by linear maps. Also, g^{-1} acts as the inverse of g, so the corresponding maps are invertible. Hence V is a representation of G. □

Because of this proposition, we will freely identify representations of G and left $\mathbb{C}[G]$-modules.

Example 5.1. *The vector space $\mathbb{C}[G]$ itself is a left $\mathbb{C}[G]$-module by left multiplication. Equivalently, G acts on $\mathbb{C}[G]$ by*

$$g\left(\sum_{s\in G} c_s s\right) = \sum_{s\in G} c_s (gs).$$

Under the identification between the basis element $s \in \mathbb{C}[G]$ and the delta function δ_s on G, this is the canonical representation introduced in Example 2.11.

Exercise 5.1. *Let $\rho : G \to GL(V)$ be a representation and let $a = \sum_{g\in G} c_g g \in \mathbb{C}[G]$. Define*

$$\rho(a) = \sum_{g\in G} c_g \rho(g).$$

Show that $\rho(ab) = \rho(a)\rho(b)$ for all $a, b \in \mathbb{C}[G]$.

5.2 Decomposition of $\mathbb{C}[G]$

Let $(\rho_1, V_1), \ldots, (\rho_h, V_h)$ be all irreducible representations of G, with characters $\chi_1, \ldots, \chi_h$ and degrees

$$n_i = \dim V_i.$$

By Proposition 3.7, as a left G-module, or equivalently as a left $\mathbb{C}[G]$-module, we have

$$\mathbb{C}[G] \cong n_1 V_1 \oplus \cdots \oplus n_h V_h.$$

Here $n_i V_i$ means the direct sum of n_i copies of V_i.
The group algebra also has a stronger decomposition as an algebra.

Theorem 5.1. *Define*

$$\Phi : \mathbb{C}[G] \to \bigoplus_{i=1}^{h} \mathrm{End}(V_i)$$

by

$$\Phi(a) = \big(\rho_1(a), \ldots, \rho_h(a)\big).$$

Then Φ is an isomorphism of algebras.

Proof. It is clear from the previous exercise that Φ is an algebra homomorphism. We first show that Φ is injective. Suppose $\Phi(a) = 0$. Then a acts by zero on every irreducible representation V_i. Since the left regular representation decomposes as

$$\mathbb{C}[G] \cong n_1 V_1 \oplus \cdots \oplus n_h V_h,$$

left multiplication by a acts as zero on $\mathbb{C}[G]$. In particular,

$$a = ae = 0,$$

where e is the identity element of G. Hence $\ker \Phi = 0$.
Now compare dimensions. The dimension of the left hand side is $|G|$. The dimension of the right hand side is

$$\sum_{i=1}^{h} \dim \mathrm{End}(V_i) = \sum_{i=1}^{h} n_i^2.$$

By Corollary 3.3,

$$\sum_{i=1}^{h} n_i^2 = |G|.$$

Therefore Φ is an injective linear map between vector spaces of the same dimension, hence it is an isomorphism. □

This theorem says that, after choosing all irreducible representations, the group algebra is the direct sum of full matrix algebras:

$$\mathbb{C}[G] \cong \mathrm{End}(V_1) \oplus \cdots \oplus \mathrm{End}(V_h).$$

In particular, all information about the irreducible representations is contained in $\mathbb{C}[G]$.

Definition 5.4. *For each irreducible representation* V_i*, define*

$$p_i = \frac{n_i}{|G|} \sum_{g \in G} \chi_i(g)^* g \in \mathbb{C}[G].$$

These elements are called the central idempotents associated with the irreducible representations.

Proposition 5.3. *The elements* p_i *satisfy*

$$p_i p_j = \delta_{ij} p_i, \qquad \sum_{i=1}^{h} p_i = e.$$

Moreover, for any representation (ρ, V)*, the operator* $\rho(p_i)$ *is the projection onto the* V_i*-isotypic component of* V*.*

Proof. The last statement is exactly Proposition 4.1 written in the language of the group algebra, because

$$\rho(p_i) = \frac{n_i}{|G|} \sum_{g \in G} \chi_i(g)^* \rho(g).$$

Apply this to the irreducible representation V_k. We get

$$\rho_k(p_i) = \begin{cases} \mathrm{Id}_{V_i}, & k = i, \\ 0, & k \neq i. \end{cases}$$

Therefore, under the algebra isomorphism Φ, p_i maps to

$$(0, \dots, 0, \mathrm{Id}_{V_i}, 0, \dots, 0).$$

The identities $p_i p_j = \delta_{ij} p_i$ and $\sum_i p_i = e$ follow immediately. □

Exercise 5.2. *Show directly from the definition that* p_i *commutes with every group element* $g \in G$*. Thus* p_i *lies in the center of* $\mathbb{C}[G]$*.*

5.3 The center of $\mathbb{C}[G]$

Definition 5.5. *Let* A *be an algebra. The center of* A *is*

$$Z(A) = \{a \in A : ab = ba, \ \forall b \in A\}.$$

Let C be a conjugacy class of G. Define the class sum

$$\Omega_C = \sum_{g \in C} g \in \mathbb{C}[G].$$

Proposition 5.4. *The class sums* Ω_C*, where* C *runs over all conjugacy classes of* G*, form a basis of* $Z(\mathbb{C}[G])$*.*

Proof. Let

$$a = \sum_{g \in G} a_g g \in \mathbb{C}[G].$$

The element a is central if and only if $hah^{-1} = a$ for all $h \in G$. But

$$hah^{-1} = \sum_{g \in G} a_g (hgh^{-1}).$$

Thus $hah^{-1} = a$ for all $h \in G$ if and only if the coefficient a_g is constant on conjugacy classes. Therefore every central element is a linear combination of class sums. Since distinct conjugacy classes are disjoint, the class sums are linearly independent. □

Corollary 5.1.

$$\dim Z(\mathbb{C}[G]) = \text{the number of conjugacy classes of } G.$$

This agrees with Theorem 3.5, which says that the number of irreducible representations is the number of conjugacy classes.

Proposition 5.5. *Let $z \in Z(\mathbb{C}[G])$. For each irreducible representation (ρ_i, V_i), there is a scalar $\omega_i(z) \in \mathbb{C}$ such that*

$$\rho_i(z) = \omega_i(z) \operatorname{Id}_{V_i} .$$

If C is a conjugacy class and $c \in C$, then

$$\omega_i(\Omega_C) = \frac{|C|\chi_i(c)}{n_i}.$$

Proof. Since z is central, for every $g \in G$ we have

$$\rho_i(g)\rho_i(z) = \rho_i(gz) = \rho_i(zg) = \rho_i(z)\rho_i(g).$$

Thus $\rho_i(z)$ commutes with the irreducible representation ρ_i. By Schur's lemma,

$$\rho_i(z) = \omega_i(z) \operatorname{Id}_{V_i}$$

for some scalar $\omega_i(z)$.
Now take $z = \Omega_C$. Then

$$\rho_i(\Omega_C) = \sum_{g \in C} \rho_i(g).$$

Taking trace on both sides gives

$$n_i\omega_i(\Omega_C) = \sum_{g \in C} \chi_i(g) = |C|\chi_i(c),$$

because χ_i is constant on conjugacy classes. Hence

$$\omega_i(\Omega_C) = \frac{|C|\chi_i(c)}{n_i}.$$

□

Definition 5.6. *The scalar $\omega_i(z)$ is called the central character of z on V_i.*

Proposition 5.6. *The elements $p_1, \dots, p_h$ form another basis of $Z(\mathbb{C}[G])$. More precisely,*

$$Z(\mathbb{C}[G]) = \mathbb{C}p_1 \oplus \cdots \oplus \mathbb{C}p_h.$$

Proof. Under the algebra isomorphism

$$\mathbb{C}[G] \cong \bigoplus_{i=1}^{h} \operatorname{End}(V_i),$$

the center of the right hand side is

$$\mathbb{C} \operatorname{Id}_{V_1} \oplus \cdots \oplus \mathbb{C} \operatorname{Id}_{V_h} .$$

The element p_i maps to the element whose i-th component is Id_{V_i} and whose other components are zero. Hence the p_i form a basis of the center. □

Exercise 5.3. *Let C and D be conjugacy classes of G. Since $\Omega_C\Omega_D$ is central, there exist numbers $N_{CD}^E \in \mathbb{C}$ such that*

$$\Omega_C\Omega_D = \sum_E N_{CD}^E \Omega_E,$$

where E runs over conjugacy classes. Show that each N_{CD}^E is in fact a nonnegative integer. Interpret N_{CD}^E as the number of ways to write a fixed element of E as a product cd with $c \in C$ and $d \in D$.

5.4 Basic properties of integers

In this section, the word "integer" means algebraic integer. These numbers generalize ordinary integers and naturally appear as character values.

Definition 5.7. *A complex number $\alpha \in \mathbb{C}$ is called an algebraic integer if there exists a monic polynomial*

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0, \qquad a_i \in \mathbb{Z},$$

such that

$$f(\alpha) = 0.$$

Every ordinary integer is an algebraic integer, since $m \in \mathbb{Z}$ is a root of $x - m$.

Proposition 5.7. *Let $\alpha \in \mathbb{C}$. Then α is an algebraic integer if and only if there exists a nonzero finitely generated abelian group $M \subset \mathbb{C}$ such that*

$$\alpha M \subset M.$$

Proof. Suppose α is an algebraic integer and satisfies

$$\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0 = 0.$$

Let $M = \mathbb{Z} + \mathbb{Z}\alpha + \cdots + \mathbb{Z}\alpha^{n-1}$, then M is finitely generated and $\alpha M \subset M$, because α^n can be written as an integer linear combination of $1, \alpha, \ldots, \alpha^{n-1}$.
Conversely, suppose M is generated by $m_1, \ldots, m_r$ as an abelian group and $\alpha M \subset M$. Then there are integers a_{ij} such that

$$\alpha m_j = \sum_{i=1}^{r} a_{ij} m_i.$$

Let $A = (a_{ij})$. Then the equations above say that the matrix $\alpha I - A$ has a nonzero vector in its kernel. Hence

$$\det(\alpha I - A) = 0.$$

The polynomial $\det(xI - A)$ is monic with integer coefficients, so α is an algebraic integer. ☐

Proposition 5.8. *The algebraic integers form a subring of $\mathbb{C}$. In other words, if α and β are algebraic integers, then $\alpha + \beta$, $\alpha\beta$, are also algebraic integers.*

Proof. Choose finitely generated abelian groups $M, N \subset \mathbb{C}$ such that

$$\alpha M \subset M, \qquad \beta N \subset N.$$

Let L be the subgroup of $\mathbb{C}$ generated by all products mn with $m \in M$ and $n \in N$. Since M and N are finitely generated, L is finitely generated. Moreover, L is stable under multiplication by α, by β, and therefore by $\alpha+\beta$ and $\alpha\beta$. By the previous proposition, $\alpha + \beta$ and $\alpha\beta$ are algebraic integers. ☐

Proposition 5.9. *If $\alpha \in \mathbb{Q}$ is an algebraic integer, then $\alpha \in \mathbb{Z}$.*

Proof. Write $\alpha = p/q$ with $p, q \in \mathbb{Z}$, $q > 0$, and $\gcd(p, q) = 1$. Suppose α satisfies

$$\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0 = 0, \qquad a_i \in \mathbb{Z}.$$

Multiplying by q^n, we get

$$p^n + a_{n-1}p^{n-1}q + \cdots + a_1 p q^{n-1} + a_0 q^n = 0.$$

Thus q divides p^n. Since $\gcd(p, q) = 1$, we must have $q = 1$. Therefore $\alpha \in \mathbb{Z}$. ☐

Proposition 5.10. *If A is a square matrix with integer entries, then every eigenvalue of A is an algebraic integer.*

Proof. Every eigenvalue of A is a root of the characteristic polynomial $\det(xI - A)$, which is monic and has integer coefficients. ☐

Exercise 5.4. *Show that every root of unity is an algebraic integer. Conclude that if ρ is a representation of a finite group, then every character value $\chi_\rho(g)$ is an algebraic integer.*

5.5 Integrality properties of characters. Applications

We now apply algebraic integers to character theory. The key point is that class sums act by algebraic integers on irreducible representations.

Proposition 5.11. *Let C be a conjugacy class of G, let $c \in C$, and let (ρ_i, V_i) be an irreducible representation of degree n_i and character χ_i. Then $\frac{|C|\chi_i(c)}{n_i}$ is an algebraic integer.*

Proof. The class sum Ω_C acts on the irreducible representation V_i by the scalar

$$\omega_i(\Omega_C) = \frac{|C|\chi_i(c)}{n_i}.$$

On the other hand, left multiplication by Ω_C on $\mathbb{C}[G]$ is represented in the basis G by a matrix with integer entries: indeed, for $s \in G$,

$$\Omega_C s = \sum_{g \in C} gs,$$

which is an integer linear combination of elements of G. Therefore every eigenvalue of this operator is an algebraic integer. Since $\omega_i(\Omega_C)$ occurs as an eigenvalue of the action of Ω_C on the regular representation, it is an algebraic integer. □

Theorem 5.2. *Let (ρ_i, V_i) be an irreducible representation of G of degree n_i. Then n_i divides $|G|$. In other words, the degree of every irreducible representation divides the order of the group.*

Proof. Using the character orthogonality relation $\langle \chi_i, \chi_i \rangle = 1$, we have

$$|G| = \sum_{g \in G} \chi_i(g)^* \chi_i(g) = \sum_{g \in G} \chi_i(g^{-1})\chi_i(g).$$

Group the sum by conjugacy classes. Let C run over conjugacy classes and choose $c \in C$. Then

$$|G| = \sum_C |C|\chi_i(c^{-1})\chi_i(c).$$

Dividing by n_i, we get

$$\frac{|G|}{n_i} = \sum_C \left(\frac{|C|\chi_i(c)}{n_i} \right) \chi_i(c^{-1}).$$

The first factor in each term is an algebraic integer by the previous proposition. The second factor is a character value, hence is also an algebraic integer. Therefore $|G|/n_i$ is an algebraic integer.
But $|G|/n_i$ is a rational number. A rational algebraic integer is an ordinary integer. Hence $|G|/n_i \in \mathbb{Z}$, so n_i divides $|G|$. □

Corollary 5.2. *If G is abelian, then every irreducible representation of G has degree 1.*

Proof. If G is abelian, every conjugacy class has one element. Hence the number of conjugacy classes is $|G|$, so G has $|G|$ irreducible representations. If their degrees are $n_1, \ldots, n_{|G|}$, then

$$\sum_{i=1}^{|G|} n_i^2 = |G|.$$

Since each $n_i \geq 1$, the only possibility is $n_i = 1$ for all i. □

Example 5.2. *Let $G = D_4$. We know that $|D_4| = 8$ and that D_4 has five conjugacy classes. Hence there are five irreducible representations, with degrees $n_1, \ldots, n_5$ satisfying*

$$n_1^2 + \cdots + n_5^2 = 8.$$

Since each $n_i \geq 1$, the only possibility is

$$1, 1, 1, 1, 2.$$

This agrees with the character table found in Section 3.6.

Proposition 5.12. *Let C, D, E be conjugacy classes and write*

$$\Omega_C \Omega_D = \sum_E N_{CD}^E \Omega_E.$$

For every irreducible representation V_i, we have

$$\omega_i(\Omega_C)\omega_i(\Omega_D) = \sum_E N_{CD}^E \omega_i(\Omega_E).$$

Equivalently,

$$\frac{|C|\chi_i(c)}{n_i} \frac{|D|\chi_i(d)}{n_i} = \sum_E N_{CD}^E \frac{|E|\chi_i(e)}{n_i},$$

where $c \in C$, $d \in D$, and $e \in E$.

Proof. Apply the irreducible representation ρ_i to the identity

$$\Omega_C \Omega_D = \sum_E N_{CD}^E \Omega_E.$$

Since each class sum is central, it acts on V_i by a scalar. This gives the desired identity. □

This proposition says that the character table diagonalizes the multiplication rules of class sums. In computations, this gives strong constraints on possible character tables.

Exercise 5.5. *For D_4, compute the products of the class sums corresponding to the five conjugacy classes in Section 3.6. Check that the character table in Table 1 diagonalizes these multiplication rules.*

6 Induced representations

6.1 Tensor product of modules

We first recall the tensor product. Tensor products are useful because induction is most naturally written as

$$\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W.$$

Definition 6.1. *Let V and W be vector spaces over $\mathbb{C}$. Their tensor product $V \otimes_{\mathbb{C}} W$ is the vector space generated by symbols $v \otimes w$, where $v \in V$ and $w \in W$, subject to the bilinear relations*

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w,$$

$$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2,$$

$$(cv) \otimes w = v \otimes (cw) = c(v \otimes w).$$

If $\{v_i\}_{i=1}^m$ is a basis of V and $\{w_j\}_{j=1}^n$ is a basis of W, then

$$\{v_i \otimes w_j : 1 \leq i \leq m,\ 1 \leq j \leq n\}$$

is a basis of $V \otimes_{\mathbb{C}} W$. Hence

$$\dim(V \otimes_{\mathbb{C}} W) = \dim V \dim W.$$

Definition 6.2. *Let V and W be G-modules. The tensor product representation $V \otimes W$ is the vector space $V \otimes_{\mathbb{C}} W$ with G-action*

$$g(v \otimes w) = (gv) \otimes (gw).$$

Proposition 6.1. *If V and W have characters χ_V and χ_W, then $V \otimes W$ has character*

$$\chi_{V \otimes W}(g) = \chi_V(g)\chi_W(g).$$

Proof. Choose bases of V and W. If the matrices of g on V and W are A and B, then the matrix of g on $V \otimes W$ is the Kronecker product $A \otimes B$. Its trace is

$$\operatorname{tr}(A \otimes B) = \operatorname{tr}(A) \operatorname{tr}(B).$$

Therefore

$$\chi_{V \otimes W}(g) = \chi_V(g)\chi_W(g).$$

□

Definition 6.3. *Let V be a G-module. The dual representation $V^\vee$ is the dual vector space of V, with action*

$$(g\alpha)(v) = \alpha(g^{-1}v), \qquad \alpha \in V^\vee,\ v \in V.$$

Exercise 6.1. *Show that*

$$\chi_{V^\vee}(g) = \chi_V(g^{-1}) = \chi_V(g)^*.$$

We will also need tensor products over an algebra.

Definition 6.4. *Let A be an algebra over $\mathbb{C}$. Let M be a right A-module and let N be a left A-module. The tensor product of M and N over A, denoted by $M \otimes_A N$, is the quotient of $M \otimes_{\mathbb{C}} N$ by the subspace generated by elements of the form*

$$(ma) \otimes n - m \otimes (an), \qquad m \in M,\ n \in N,\ a \in A.$$

The image of $m \otimes n$ in the quotient is denoted by $m \otimes_A n$.

Thus, in $M \otimes_A N$, we impose the relation

$$ma \otimes_A n = m \otimes_A an.$$

Example 6.1. *Let $H \subset G$ be a subgroup. Then $\mathbb{C}[G]$ is a left $\mathbb{C}[G]$-module by left multiplication and a right $\mathbb{C}[H]$-module by right multiplication. These two actions commute. Hence, if W is an H-module, the vector space*

$$\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$$

is naturally a left $\mathbb{C}[G]$-module.

6.2 Induced representations

Definition 6.5. *Let $H \subset G$ be a subgroup and let W be an H-module. The induced representation of W from H to G is*

$$\operatorname{Ind}_H^G W = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W.$$

The action of G is given by left multiplication on the first factor:

$$g'(g \otimes w) = (g'g) \otimes w.$$

Intuitively, induction takes a representation of a subgroup and makes copies of it over the cosets of the subgroup. This is why induced representations naturally appear when a state is first defined with a smaller symmetry group and then moved around by the full group.

Proposition 6.2. *Let $r_1, \ldots, r_m$ be representatives of the left cosets G/H, so*

$$G = r_1 H \sqcup \cdots \sqcup r_m H.$$

Then every element of $\operatorname{Ind}_H^G W$ can be written uniquely as

$$\sum_{i=1}^{m} r_i \otimes w_i, \qquad w_i \in W.$$

In particular,

$$\dim \operatorname{Ind}_H^G W = [G : H] \dim W.$$

Moreover, if $g \in G$ and $gr_i = r_j h$ with $h \in H$, then

$$g(r_i \otimes w) = r_j \otimes (hw).$$

Proof. Every element of G can be written uniquely as $r_i h$ for some i and $h \in H$. In the tensor product over $\mathbb{C}[H]$, we have

$$r_i h \otimes w = r_i \otimes hw.$$

Therefore every element of $\operatorname{Ind}_H^G W$ can be written as $\sum_i r_i \otimes w_i$.
Uniqueness follows from the vector space decomposition

$$\mathbb{C}[G] = \bigoplus_{i=1}^{m} r_i \mathbb{C}[H]$$

as a right $\mathbb{C}[H]$-module. Hence

$$\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W \cong \bigoplus_{i=1}^{m} r_i \mathbb{C}[H] \otimes_{\mathbb{C}[H]} W \cong \bigoplus_{i=1}^{m} W.$$

The formula for the action follows from

$$g(r_i \otimes w) = (gr_i) \otimes w = (r_j h) \otimes w = r_j \otimes hw.$$

□

Example 6.2. *If $H = G$, then*

$$\operatorname{Ind}_G^G W \cong W.$$

If $H = \{e\}$ and $W = \mathbb{C}$ is the trivial representation, then

$$\operatorname{Ind}_{\{e\}}^G \mathbb{C} \cong \mathbb{C}[G],$$

the canonical representation.

Example 6.3. *Let $\mathbb{C}$ be the trivial representation of H. Then*

$$\operatorname{Ind}_H^G \mathbb{C}$$

is the permutation representation of G on the set of left cosets G/H. Indeed, with coset representatives r_i, the basis vector $r_i \otimes 1$ corresponds to the coset $r_i H$, and G permutes these cosets by left multiplication.

Proposition 6.3 (Function model)**.** *The induced representation* $\operatorname{Ind}_H^G W$ *is isomorphic to the vector space of functions*

$$f : G \to W$$

satisfying

$$f(xh) = h^{-1} f(x), \qquad x \in G,\ h \in H.$$

The action of G is

$$(gf)(x) = f(g^{-1}x).$$

Proof. Choose left coset representatives $r_1, \dots, r_m$. A function satisfying $f(xh) = h^{-1}f(x)$ is uniquely determined by its values $f(r_i)$. Hence the function space is naturally isomorphic to $\bigoplus_i W$, just like $\operatorname{Ind}_H^G W$. Under this identification, $r_i \otimes w_i$ corresponds to the function defined by

$$f(r_i h) = h^{-1} w_i, \qquad h \in H,$$

and equal to zero on the other cosets. The formula $(gf)(x) = f(g^{-1}x)$ agrees with left multiplication on the tensor product. ☐

Exercise 6.2. *Let* $K \subset H \subset G$ *be subgroups and let* U *be a* K*-module. Show that*

$$\operatorname{Ind}_H^G \left(\operatorname{Ind}_K^H U \right) \cong \operatorname{Ind}_K^G U.$$

This is called transitivity of induction.

6.3 The character of an induced representation; the reciprocity formula

Let ψ be the character of an H-module W. We now compute the character of $\operatorname{Ind}_H^G W$.

Theorem 6.1. *Let* $H \subset G$ *and let* W *be an* H*-module with character* ψ*. The character of* $\operatorname{Ind}_H^G W$ *is*

$$\chi_{\operatorname{Ind}_H^G W}(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1}gx \in H}} \psi(x^{-1}gx).$$

Proof. Choose representatives $r_1, \dots, r_m$ of the left cosets G/H. By the previous proposition,

$$\operatorname{Ind}_H^G W \cong \bigoplus_{i=1}^{m} r_i \otimes W.$$

The element $g \in G$ sends the summand $r_i \otimes W$ to the summand corresponding to the coset gr_iH.
Only those summands that are mapped to themselves contribute to the trace. The summand $r_i \otimes W$ is fixed precisely when

$$gr_iH = r_iH,$$

equivalently $r_i^{-1}gr_i \in H$. On such a fixed summand, the action is

$$g(r_i \otimes w) = r_i \otimes \big((r_i^{-1}gr_i)w \big),$$

so the contribution to the trace is $\psi(r_i^{-1}gr_i)$. Hence

$$\chi_{\operatorname{Ind}_H^G W}(g) = \sum_{\substack{i \\ r_i^{-1}gr_i \in H}} \psi(r_i^{-1}gr_i).$$

Replacing the sum over coset representatives by a sum over all $x \in G$ gives

$$\chi_{\operatorname{Ind}_H^G W}(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1}gx \in H}} \psi(x^{-1}gx),$$

because each left coset contains $|H|$ elements and ψ is a class function on H. ☐

Definition 6.6. *If α and β are class functions on G, write*

$$\langle \alpha, \beta \rangle_G = \frac{1}{|G|} \sum_{g \in G} \alpha(g)^* \beta(g).$$

If ψ is a class function on H, define the induced class function $\operatorname{Ind}_H^G \psi$ by the same formula:

$$(\operatorname{Ind}_H^G \psi)(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1} g x \in H}} \psi(x^{-1} g x).$$

Definition 6.7. *Let V be a G-module. The restriction of V to H is the same vector space V, viewed only as an H-module. It is denoted by $\operatorname{Res}_H^G V$. If χ is the character of V, then the character of $\operatorname{Res}_H^G V$ is simply $\chi|_H$.*

Theorem 6.2 (Frobenius reciprocity)**.** *Let W be an H-module with character ψ, and let V be a G-module with character χ. Then*

$$\left\langle \operatorname{Ind}_H^G \psi, \chi \right\rangle_G = \left\langle \psi, \operatorname{Res}_H^G \chi \right\rangle_H.$$

Proof. Using the character formula for induction,

$$\left\langle \operatorname{Ind}_H^G \psi, \chi \right\rangle_G = \frac{1}{|G|} \sum_{g \in G} \left(\frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1} g x \in H}} \psi(x^{-1} g x) \right)^* \chi(g).$$

We may rewrite this as a sum over pairs $(x, h) \in G \times H$ by setting

$$h = x^{-1} g x, \qquad \text{or equivalently} \qquad g = x h x^{-1}.$$

Thus

$$\left\langle \operatorname{Ind}_H^G \psi, \chi \right\rangle_G = \frac{1}{|G||H|} \sum_{x \in G} \sum_{h \in H} \psi(h)^* \chi(x h x^{-1}).$$

Since χ is a class function on G, $\chi(x h x^{-1}) = \chi(h)$. Hence

$$\left\langle \operatorname{Ind}_H^G \psi, \chi \right\rangle_G = \frac{1}{|H|} \sum_{h \in H} \psi(h)^* \chi(h) = \left\langle \psi, \operatorname{Res}_H^G \chi \right\rangle_H.$$

□

Proposition 6.4 (Frobenius reciprocity, module form)**.** *There is a natural isomorphism*

$$\operatorname{Hom}_G(\operatorname{Ind}_H^G W, V) \cong \operatorname{Hom}_H(W, \operatorname{Res}_H^G V).$$

Proof. Let $F : \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W \to V$ be a G-module homomorphism. Define

$$\phi_F : W \to V, \qquad \phi_F(w) = F(e \otimes w).$$

For $h \in H$, we have

$$\phi_F(hw) = F(e \otimes hw) = F(h \otimes w) = hF(e \otimes w) = h\phi_F(w),$$

so ϕ_F is an H-module homomorphism.
Conversely, given an H-module homomorphism $\phi : W \to \operatorname{Res}_H^G V$, define

$$F_\phi(g \otimes w) = g\phi(w).$$

This is well-defined because

$$F_\phi(gh \otimes w) = gh\phi(w) = g\phi(hw) = F_\phi(g \otimes hw).$$

It is also clearly G-equivariant. The two constructions are inverse to each other. □

Exercise 6.3. *Let 1_H be the trivial character of H. Show that*

$$\left\langle \operatorname{Ind}_H^G 1_H, \chi \right\rangle_G = \frac{1}{|H|} \sum_{h \in H} \chi(h).$$

Interpret the right hand side as the dimension of the H-invariant subspace of V.

6.4 Restriction to subgroups

Let $K \subset G$ be another subgroup. In many applications one first induces a representation from H to G, and then restricts it to K. The answer is controlled by double cosets.

Definition 6.8. *Let $H, K \subset G$ be subgroups. A double coset is a subset of G of the form*

$$KxH = \{kxh : k \in K,\ h \in H\}.$$

The set of double cosets is denoted by

$$K\backslash G/H.$$

For $x \in G$, write

$${}^{x}H = xHx^{-1}.$$

If W is an H-module, define the conjugate representation ${}^{x}W$ of ${}^{x}H$ as follows: the underlying vector space is still W, and

$$(xhx^{-1})w = hw, \qquad h \in H,\ w \in W.$$

Theorem 6.3 (Mackey restriction formula)**.** *Let $H, K \subset G$ and let W be an H-module. For each representative x of a double coset in $K\backslash G/H$, set*

$$K_x = K \cap xHx^{-1}.$$

Then

$$\operatorname{Res}^G_K \operatorname{Ind}^G_H W \cong \bigoplus_{x \in K\backslash G/H} \operatorname{Ind}^K_{K_x} \left(\operatorname{Res}^{xHx^{-1}}_{K_x} ({}^{x}W) \right).$$

Proof. As a $(\mathbb{C}[K], \mathbb{C}[H])$-bimodule, we have the direct sum decomposition

$$\mathbb{C}[G] = \bigoplus_{x \in K\backslash G/H} \mathbb{C}[KxH],$$

where x runs over a choice of double coset representatives. Therefore

$$\operatorname{Res}^G_K \operatorname{Ind}^G_H W = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W \cong \bigoplus_{x \in K\backslash G/H} \mathbb{C}[KxH] \otimes_{\mathbb{C}[H]} W.$$

Fix one double coset representative x and put $K_x = K \cap xHx^{-1}$. We claim that

$$\mathbb{C}[KxH] \otimes_{\mathbb{C}[H]} W \cong \mathbb{C}[K] \otimes_{\mathbb{C}[K_x]} \operatorname{Res}^{xHx^{-1}}_{K_x} ({}^{x}W).$$

Define a map from the right hand side to the left hand side by

$$k \otimes w \mapsto kx \otimes w.$$

This is well-defined. Indeed, if $l \in K_x$, then $l = xhx^{-1}$ for some $h \in H$. In the right hand side, the tensor relation identifies

$$kl \otimes w = k \otimes lw = k \otimes hw.$$

In the left hand side,

$$klx \otimes w = kxh \otimes w = kx \otimes hw.$$

Thus the map respects the tensor relations. It is K-equivariant, and one checks from the description of the double coset KxH that it is bijective. Summing over double cosets gives the formula. ☐

Corollary 6.1 (Branching rule)**.** *Let V_i be an irreducible representation of G with character χ_i, and let W_α be an irreducible representation of H with character ψ_α. The multiplicity of W_α in $\operatorname{Res}^G_H V_i$ is*

$$\left\langle \psi_\alpha, \operatorname{Res}^G_H \chi_i \right\rangle_H .$$

By Frobenius reciprocity, this is also the multiplicity of V_i in $\operatorname{Ind}^G_H W_\alpha$:

$$\left\langle \psi_\alpha, \operatorname{Res}^G_H \chi_i \right\rangle_H = \left\langle \operatorname{Ind}^G_H \psi_\alpha, \chi_i \right\rangle_G .$$

Exercise 6.4. *Let $H \subset G$ and let V be a G-module. Show that the multiplicity of the trivial representation of H in $\operatorname{Res}^G_H V$ is the dimension of the subspace*

$$V^H = \{v \in V : hv = v,\ \forall h \in H\}.$$

Use Frobenius reciprocity to reinterpret this multiplicity inside $\operatorname{Ind}^G_H \mathbb{C}$.

6.5 Mackey's irreducibility criterion

We now use the Mackey restriction formula to decide when an induced representation is irreducible.

Theorem 6.4 (Mackey's irreducibility criterion)**.** *Let $H \subset G$ and let W be an irreducible H-module. For $x \in G$, define*

$$H_x = H \cap xHx^{-1}.$$

Then $\operatorname{Ind}_H^G W$ is irreducible if and only if, for every $x \notin H$,

$$\operatorname{Hom}_{H_x}\left(\operatorname{Res}^H_{H_x} W, \operatorname{Res}^{xHx^{-1}}_{H_x}({}^xW)\right) = 0.$$

Equivalently, for every nontrivial double coset $HxH \neq H$, the two H_x-modules

$$\operatorname{Res}^H_{H_x} W, \qquad \operatorname{Res}^{xHx^{-1}}_{H_x}({}^xW)$$

have no irreducible component in common.

Proof. Let $U = \operatorname{Ind}_H^G W$. By Theorem 3.3, U is irreducible if and only if

$$\langle \chi_U, \chi_U \rangle_G = 1.$$

By Frobenius reciprocity,

$$\langle \chi_U, \chi_U \rangle_G = \left\langle \chi_W, \operatorname{Res}^G_H \chi_U \right\rangle_H.$$

Now apply the Mackey restriction formula with $K = H$:

$$\operatorname{Res}^G_H \operatorname{Ind}^G_H W \cong \bigoplus_{x \in H\backslash G/H} \operatorname{Ind}^H_{H_x}\left(\operatorname{Res}^{xHx^{-1}}_{H_x}({}^xW)\right).$$

Therefore

$$\langle \chi_U, \chi_U \rangle_G = \sum_{x \in H\backslash G/H} \left\langle \operatorname{Res}^H_{H_x} W, \operatorname{Res}^{xHx^{-1}}_{H_x}({}^xW)\right\rangle_{H_x},$$

where we used Frobenius reciprocity again for each summand. The double coset H itself corresponds to $x = e$. Its contribution is

$$\langle \chi_W, \chi_W \rangle_H = 1,$$

because W is irreducible. Every other summand is a nonnegative integer, since it is the dimension of a space of homomorphisms, or equivalently the number of common irreducible components counted with multiplicity. Hence

$$\langle \chi_U, \chi_U \rangle_G = 1$$

if and only if every contribution from a nontrivial double coset is zero. This is exactly the stated condition. □

Corollary 6.2. *Suppose H is a normal subgroup of G, and let W be an irreducible H-module. Then $\operatorname{Ind}_H^G W$ is irreducible if and only if ${}^xW \not\cong W$ as H-modules for every $x \notin H$.*

Proof. If H is normal, then $xHx^{-1} = H$ and $H_x = H$ for every $x \in G$. Mackey's criterion becomes

$$\operatorname{Hom}_H(W, {}^xW) = 0$$

for every $x \notin H$. Since W and ${}^xW$ are both irreducible, Schur's lemma says this is equivalent to $W \not\cong {}^xW$. □

Exercise 6.5. *Let $H = \{e\}$. Use Mackey's criterion to show that the canonical representation $\mathbb{C}[G]$ is irreducible if and only if G is the trivial group.*

Exercise 6.6. *Assume H is normal in G. Let W be an irreducible representation of H, and let*

$$G_W = \{g \in G : {}^gW \cong W\}.$$

Show that G_W is a subgroup of G containing H. This subgroup is called the inertia group of W and will be useful when studying normal subgroups and semidirect products.

7 Examples of induced representations

In the previous chapter we introduced induction in an abstract form:

$$\mathrm{Ind}_H^G W = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W.$$

In this chapter we discuss two important situations where induced representations can be described rather explicitly. The first one is the case of a normal subgroup. The second one is the case of a semidirect product by an abelian group. The second case is the finite-group version of the little group method used for space groups in solid state physics.

7.1 Normal subgroups; applications to the degrees of the irreducible representations

Let N be a normal subgroup of G. If W is a representation of N and $g \in G$, then we can construct another representation of N by conjugating the action.

Definition 7.1. *Let N be a normal subgroup of G, and let W be an N-module. For $g \in G$, the conjugate N-module ${}^gW$ is the same vector space W, with N-action defined by*

$$n \cdot w = (g^{-1}ng)w, \qquad n \in N,\ w \in W.$$

If ψ is the character of W, then the character of ${}^gW$ is

$$({}^g\psi)(n) = \psi(g^{-1}ng).$$

Since N is normal, $g^{-1}ng \in N$, so the definition makes sense. Thus G acts on the set of isomorphism classes of irreducible representations of N.

Definition 7.2. *Let W be an irreducible representation of N. The inertia group of W is*

$$G_W = \{g \in G : {}^gW \cong W\}.$$

Equivalently, if ψ is the character of W, then

$$G_W = \{g \in G : {}^g\psi = \psi\}.$$

The inertia group is the subgroup of G that preserves the irreducible representation W of N. It always contains N, because conjugation by an element of N does not change the isomorphism class of an irreducible representation of N.

Proposition 7.1. *Let N be a normal subgroup of G. Let V be an irreducible representation of G. Let W be an irreducible representation of N that appears in $\mathrm{Res}_N^G V$. Then there exists a positive integer e such that*

$$\mathrm{Res}_N^G V \cong e \bigoplus_{i=1}^{r} {}^{g_i}W,$$

where $g_1, \dots, g_r$ are representatives of G/G_W. In particular,

$$\dim V = e[G : G_W] \dim W.$$

Proof. Since N is finite, $\mathrm{Res}_N^G V$ is completely reducible as an N-module. For an irreducible N-module U, let $V[U]$ be the sum of all N-submodules of V isomorphic to U. This is the U-isotypic component of $\mathrm{Res}_N^G V$. We claim that for $g \in G$,

$$gV[U] = V[{}^gU].$$

Indeed, if $u \in U$ and $n \in N$, then

$$n(gu) = g(g^{-1}ng)u.$$

Thus the N-module generated by gu is the conjugate of the N-module generated by u.
Now consider the direct sum of all isotypic components in the G-orbit of W:

$$X = \bigoplus_{g \in G/G_W} V[{}^gW].$$

The previous paragraph shows that X is stable under the action of G. It is nonzero because W appears in V. Since V is irreducible as a G-module, we must have $X = V$.

Finally, the map $v \mapsto gv$ gives a vector space isomorphism from $V[W]$ to $V[{}^gW]$. Hence all these isotypic components contain the same number e of copies of the corresponding irreducible N-module. Therefore

$$\operatorname{Res}_N^G V \cong e \bigoplus_{i=1}^{r} {}^{g_i}W.$$

Taking dimensions gives

$$\dim V = e[G : G_W] \dim W.$$

□

Corollary 7.1. *Let χ be the character of V and let ψ be the character of W. Under the assumptions of the previous proposition,*

$$\chi|_N = e \sum_{i=1}^{r} {}^{g_i}\psi.$$

In other words,

$$\chi(n) = e \sum_{i=1}^{r} \psi(g_i^{-1} n g_i), \qquad n \in N.$$

Proof. Characters add under direct sums, so this follows directly from the previous proposition. □

Proposition 7.2. *Let N be a normal subgroup of G, and let W be an irreducible representation of N. Then $\operatorname{Ind}_N^G W$ is irreducible if and only if $G_W = N$. Equivalently, $\operatorname{Ind}_N^G W$ is irreducible if and only if ${}^gW \not\cong W$ for every $g \notin N$.*

Proof. This is the normal-subgroup corollary of Mackey's irreducibility criterion from the previous chapter. Since N is normal, for every $g \in G$ we have $gNg^{-1} = N$ and $N \cap gNg^{-1} = N$. Therefore Mackey's criterion becomes

$$\operatorname{Hom}_N(W, {}^gW) = 0$$

for every $g \notin N$. By Schur's lemma, this is equivalent to ${}^gW \not\cong W$ for every $g \notin N$, i.e. $G_W = N$. □

Proposition 7.3. *Let N be a normal subgroup of G, let W be an irreducible representation of N, and let V be an irreducible representation of G. Then the multiplicity of V in $\operatorname{Ind}_N^G W$ is equal to the multiplicity of W in $\operatorname{Res}_N^G V$:*

$$\left\langle \chi_{\operatorname{Ind}_N^G W}, \chi_V \right\rangle_G = \langle \chi_W, \chi_V|_N \rangle_N .$$

In particular, if W appears in $\operatorname{Res}_N^G V$ with multiplicity e, then V appears in $\operatorname{Ind}_N^G W$ with multiplicity e.

Proof. This is Frobenius reciprocity applied to the subgroup $N \subset G$. □

This proposition is often useful in practice. Instead of constructing all irreducible representations of G directly, one first studies the usually simpler irreducible representations of a normal subgroup N, then induces them up to G.

Corollary 7.2. *Let N be a normal subgroup of G, and let W be an irreducible representation of N. If $G_W = N$, then $\operatorname{Ind}_N^G W$ is an irreducible representation of G of degree $[G : N] \dim W$.*

Proof. The irreducibility follows from the previous proposition. The dimension formula follows from the general formula

$$\dim \operatorname{Ind}_N^G W = [G : N] \dim W.$$

□

The following theorem is a standard application of the same ideas. It is very useful as a quick check on possible character tables.

Theorem 7.1 (Ito's theorem)**.** *Let A be an abelian normal subgroup of G. Then the degree of every irreducible representation of G divides $[G : A]$.*

Proof. We give the main idea, since the full proof naturally uses projective representations. Let V be an irreducible representation of G. Since A is abelian, every irreducible representation of A is one-dimensional. Choose a character $\lambda : A \to \mathbb{C}^\times$ that appears in $\operatorname{Res}_A^G V$, and let

$$G_\lambda = \{g \in G : {}^g\lambda = \lambda\}$$

be its inertia group. By the first proposition of this section,

$$\dim V = e[G : G_\lambda]$$

for some positive integer e. The λ-isotypic component of V carries an irreducible projective representation of the quotient G_λ/A, and e is its degree. The projective version of the divisibility theorem proved in Chapter 5 says that e divides $|G_\lambda/A| = [G_\lambda : A]$. Therefore $\dim V = e[G : G_\lambda]$ divides $[G_\lambda : A][G : G_\lambda] = [G : A]$. □

Exercise 7.1. *Prove Ito's theorem in the special case where the character λ of A extends to an ordinary one-dimensional representation of G_λ. This special case is enough for the semidirect products considered in the next section.*

Example 7.1. *Let $D_4 = \langle r, s : r^4 = e,\ s^2 = e,\ srs^{-1} = r^{-1}\rangle$. The subgroup $A = \langle r\rangle \cong C_4$ is abelian and normal. Its irreducible characters are*

$$\lambda_m(r) = \exp\left(\frac{2\pi i m}{4}\right), \qquad m = 0, 1, 2, 3.$$

Conjugation by s sends λ_m to λ_{-m}. Hence the orbits are

$$\{\lambda_0\}, \qquad \{\lambda_2\}, \qquad \{\lambda_1, \lambda_3\}.$$

For λ_1, the inertia group is exactly A. Therefore $\operatorname{Ind}_A^{D_4} \lambda_1$ is irreducible of degree $[D_4 : A] = 2$. Its character is

$$\chi(e) = 2, \qquad \chi(r^2) = -2, \qquad \chi(r) = \chi(r^3) = 0,$$

and it vanishes on the reflections. This is the two-dimensional irreducible representation in the character table of D_4.

Exercise 7.2. *Continue the previous example. Show that λ_0 and λ_2 each extend to two one-dimensional representations of D_4. Deduce the four one-dimensional irreducible representations of D_4.*

7.2 Semidirect products by an abelian group

We now study a class of groups that occurs constantly in physics. Let A be an abelian normal subgroup and let H be a subgroup acting on A by automorphisms. Suppose $G = A \rtimes H$. Thus every element of G can be written uniquely as ah, where $a \in A$ and $h \in H$. The multiplication is determined by the action of H on A:

$$(a_1h_1)(a_2h_2) = a_1(h_1a_2h_1^{-1})h_1h_2.$$

Definition 7.3. *The dual group of A is $\widehat{A} = \operatorname{Hom}(A, \mathbb{C}^\times)$. The elements of $\widehat{A}$ are the irreducible characters of A.*

Since A is abelian, all irreducible representations of A are one-dimensional, so $\widehat{A}$ is the full set of irreducible representations of A. The group H acts on $\widehat{A}$ by

$$(h\lambda)(a) = \lambda(h^{-1}ah), \qquad h \in H,\ \lambda \in \widehat{A},\ a \in A.$$

Definition 7.4. *For $\lambda \in \widehat{A}$, the little group, or stabilizer, of λ in H is*

$$H_\lambda = \{h \in H : h\lambda = \lambda\}.$$

Equivalently,

$$\lambda(h^{-1}ah) = \lambda(a), \qquad a \in A,\ h \in H_\lambda.$$

The subgroup of G associated with λ is $G_\lambda = A \rtimes H_\lambda$. The character λ extends from A to a one-dimensional representation of G_λ by making H_λ act trivially.

Proposition 7.4. *Define $\widetilde{\lambda} : G_\lambda \to \mathbb{C}^\times$ by $\widetilde{\lambda}(ah) = \lambda(a), a \in A, h \in H_\lambda$. Then $\widetilde{\lambda}$ is a one-dimensional representation of G_λ.*

Proof. Let $a_1, a_2 \in A$ and $h_1, h_2 \in H_\lambda$. Then $(a_1h_1)(a_2h_2) = a_1(h_1a_2h_1^{-1})h_1h_2$. Since $h_1 \in H_\lambda$, $\lambda(h_1a_2h_1^{-1}) = \lambda(a_2)$. Therefore

$$\widetilde{\lambda}((a_1h_1)(a_2h_2)) = \lambda(a_1)\lambda(a_2) = \widetilde{\lambda}(a_1h_1)\widetilde{\lambda}(a_2h_2).$$

□

Now let σ be an irreducible representation of H_λ. We also view σ as a representation of $G_\lambda = A \rtimes H_\lambda$ by letting A act trivially: $\sigma(ah) = \sigma(h)$. Then $\widetilde{\lambda} \otimes \sigma$ is an irreducible representation of G_λ.

Theorem 7.2 (Little group construction for $A \rtimes H$)**.** *Let $G = A \rtimes H$, where A is finite abelian. For each H-orbit in $\widehat{A}$, choose one representative λ. For each irreducible representation σ of H_λ, define*

$$V_{\lambda,\sigma} = \mathrm{Ind}_{A \rtimes H_\lambda}^{A \rtimes H}(\widetilde{\lambda} \otimes \sigma).$$

Then:

1. *$V_{\lambda,\sigma}$ is irreducible.*
2. *Every irreducible representation of G is isomorphic to one of the $V_{\lambda,\sigma}$.*
3. *$V_{\lambda,\sigma} \cong V_{\lambda',\sigma'}$ if and only if λ' lies in the H-orbit of λ and σ' is the corresponding conjugate representation of the corresponding little group.*

Moreover,

$$\dim V_{\lambda,\sigma} = [H : H_\lambda] \dim \sigma.$$

Proof. First we prove irreducibility. Put $K = A \rtimes H_\lambda$ and $W = \widetilde{\lambda} \otimes \sigma$. By Mackey's irreducibility criterion, it is enough to show that for every $x \notin K$, the representations W and ${}^xW$ have no common irreducible component after restriction to $K \cap xKx^{-1}$.

Every double coset of K in G has a representative in H, so take $x \in H$. If $x \notin H_\lambda$, then $x\lambda \neq \lambda$. The subgroup $K \cap xKx^{-1}$ contains A, and on A the representation W has character λ, while ${}^xW$ has character $x\lambda$. Since these characters of A are different, the two restricted representations cannot have a common irreducible component. Thus Mackey's criterion implies that $V_{\lambda,\sigma}$ is irreducible.

Now let V be an irreducible representation of G. Restrict V to A. Since A is abelian, we can decompose V into A-weight spaces:

$$V = \bigoplus_{\mu \in \widehat{A}} V_\mu, \qquad V_\mu = \{v \in V : av = \mu(a)v, \ \forall a \in A\}.$$

Only finitely many V_μ are nonzero. Choose λ with $V_\lambda \neq 0$. Since A is normal, the action of H sends V_μ to $V_{h\mu}$. Since V is irreducible as a G-module, the nonzero weights form a single H-orbit.

The space V_λ is stable under H_λ, hence under $A \rtimes H_\lambda$. On A, it transforms by the scalar character λ. Thus as a representation of $A \rtimes H_\lambda$, $V_\lambda \cong \widetilde{\lambda} \otimes \sigma$ for some representation σ of H_λ. If σ were reducible, then the sum of the G-translates of a proper H_λ-subrepresentation of V_λ would be a proper nonzero G-subrepresentation of V. Hence σ is irreducible.

The natural map

$$\mathrm{Ind}_{A \rtimes H_\lambda}^{G} V_\lambda \to V, \qquad x \otimes v \mapsto xv,$$

is a nonzero G-homomorphism. The source is irreducible by the first part, and the target is irreducible by assumption, so the map is an isomorphism. This proves that every irreducible representation arises in the stated way.

The statement about isomorphism classes follows from the fact that changing λ within its H-orbit only changes the choice of the weight space used to start the construction. The dimension formula follows from the general dimension formula for induction:

$$\dim V_{\lambda,\sigma} = [G : A \rtimes H_\lambda] \dim(\widetilde{\lambda} \otimes \sigma) = [H : H_\lambda] \dim \sigma.$$

□

Corollary 7.3. *As a representation of A, we have*

$$\mathrm{Res}_A^G V_{\lambda,\sigma} \cong (\dim \sigma) \bigoplus_{\mu \in H\lambda} \mu.$$

Therefore, for $a \in A$,

$$\chi_{\lambda,\sigma}(a) = (\dim \sigma) \sum_{\mu \in H\lambda} \mu(a).$$

Proof. The induced representation has one copy of the space of σ over each coset in H/H_λ. These cosets are naturally identified with the orbit $H\lambda$. On the coset corresponding to $h\lambda$, the subgroup A acts by the character $h\lambda$. Hence each weight in the orbit appears with multiplicity $\dim \sigma$. $\square$

Example 7.2 (The dihedral group)**.** *Let $D_n = C_n \rtimes C_2$, where $C_n = \langle r \rangle$ and the nontrivial element $s \in C_2$ acts by $srs^{-1} = r^{-1}$. The characters of C_n are*

$$\lambda_m(r) = \exp\left(\frac{2\pi i m}{n}\right), \qquad m = 0, 1, \ldots, n-1.$$

The action of s sends λ_m to λ_{-m}. Thus most orbits have two elements: $\{\lambda_m, \lambda_{-m}\}$. For such an orbit, $H_{\lambda_m} = \{e\}$, and the little group construction gives a two-dimensional irreducible representation $\mathrm{Ind}_{C_n}^{D_n} \lambda_m$. If $m = 0$, and also if n is even and $m = n/2$, the character λ_m is fixed by C_2. These fixed characters produce one-dimensional representations after choosing an irreducible representation of the stabilizer C_2.

Exercise 7.3. *Use the previous example with $n = 4$ to recover all five irreducible representations of D_4 and compare the answer with the character table in Section 3.6.*

Exercise 7.4. *Let $G = A \rtimes H$ with A abelian. Prove directly from the little group construction that every irreducible degree of G divides $|H| = [G : A]$.*

Exercise 7.5. *Let A be the finite translation group of a one-dimensional periodic chain with N sites, so $A \cong C_N$. Let $H = C_2$ act by inversion. Describe the H-orbits in $\widehat{A}$ and identify which momenta have nontrivial little group.*

8 Applications in solid state physics

In the first seven chapters we mainly worked with finite groups. Crystals force one new ingredient: the translation group of an infinite crystal is infinite. We will therefore use one standard result from harmonic analysis, namely Pontryagin duality for locally compact abelian groups, as an input. We will not prove this result here. The rest of the chapter returns quickly to finite groups: point groups, little co-groups, finite magnetic point groups, and finite symmetry groups of mechanical frameworks.

8.1 Translation group and the Brillouin zone

Let L be a Bravais lattice in $\mathbb{R}^d$:

$$L = \mathbb{Z}a_1 \oplus \cdots \oplus \mathbb{Z}a_d,$$

where $a_1, \ldots, a_d$ are linearly independent primitive lattice vectors. For $R \in L$, let t_R denote translation by R. The translation group is

$$T = \{t_R : R \in L\} \cong L \cong \mathbb{Z}^d,$$

with multiplication

$$t_R t_{R'} = t_{R+R'}.$$

Thus T is an infinite discrete abelian group. This is the first essential new feature, compared with the finite groups of the previous chapters. In a finite group the regular representation decomposes as a finite direct sum of irreducible representations. For the infinite translation group of a crystal, the analogous decomposition is a Fourier decomposition over a compact continuum of characters.

Definition 8.1. *The reciprocal lattice is*

$$L^* = \{K \in \mathbb{R}^d : e^{iK\cdot R} = 1,\ \forall R \in L\}.$$

Equivalently, if $b_1, \ldots, b_d$ is the reciprocal basis defined by

$$a_i \cdot b_j = 2\pi\delta_{ij},$$

then $L^* = \mathbb{Z}b_1 \oplus \cdots \oplus \mathbb{Z}b_d$.

The factor 2π is a convention from physics. It is chosen so that the plane wave $e^{ik\cdot x}$ is unchanged under $x \mapsto x + R$ precisely when $k \in L^*$.

Theorem 8.1 (Coordinate description of the reciprocal lattice)**.** *Let $K \in \mathbb{R}^d$. Write*

$$K = \alpha_1 b_1 + \cdots + \alpha_d b_d.$$

Then $K \in L^*$ if and only if $\alpha_1, \ldots, \alpha_d \in \mathbb{Z}$. Consequently

$$L^* = \mathbb{Z}b_1 \oplus \cdots \oplus \mathbb{Z}b_d.$$

Proof. If $R = n_1 a_1 + \cdots + n_d a_d$, then

$$K \cdot R = \left(\sum_{j=1}^{d} \alpha_j b_j\right) \cdot \left(\sum_{i=1}^{d} n_i a_i\right) = 2\pi \sum_{j=1}^{d} \alpha_j n_j.$$

If every α_j is an integer, then $K \cdot R$ is an integer multiple of 2π, and hence $e^{iK\cdot R} = 1$ for all $R \in L$. Conversely, if $e^{iK\cdot R} = 1$ for all $R \in L$, then in particular $e^{iK\cdot a_j} = 1$ for every j. But $K \cdot a_j = 2\pi\alpha_j$, so $\alpha_j \in \mathbb{Z}$. Thus K is an integral linear combination of $b_1, \ldots, b_d$. □

Theorem 8.2 (Characters of the translation lattice)**.** *Let $L \cong \mathbb{Z}^d$ be regarded as a discrete locally compact abelian group. Its Pontryagin dual is*

$$\widehat{L} = \mathrm{Hom}(L, U(1)) \cong (U(1))^d.$$

After choosing an embedding $L \subset \mathbb{R}^d$, this dual group can be identified with the quotient torus $\widehat{L} \cong \mathbb{R}^d/L^*$. Explicitly, every character of L is of the form

$$\chi_k(R) = e^{-ik\cdot R}, \qquad k \in \mathbb{R}^d/L^*.$$

This is the lattice case of Pontryagin duality for locally compact abelian groups [16, 11].

Proof. Because L is discrete, every homomorphism $L \to U(1)$ is continuous. A character $\chi : L \to U(1)$ is determined by its values on the basis vectors $a_1, \ldots, a_d$. Write $\chi(a_j) = e^{-i\theta_j}, \theta_j \in \mathbb{R}/2\pi\mathbb{Z}$. Then for $R = n_1 a_1 + \cdots + n_d a_d$ we have

$$\chi(R) = \prod_{j=1}^{d} \chi(a_j)^{n_j} = e^{-i(n_1\theta_1 + \cdots + n_d\theta_d)}.$$

Now set $k = \frac{\theta_1}{2\pi} b_1 + \cdots + \frac{\theta_d}{2\pi} b_d$. Since $k \cdot a_j = \theta_j$, this gives $\chi(R) = e^{-ik\cdot R}$. Changing θ_j by $2\pi m_j$ changes k by

$$m_1 b_1 + \cdots + m_d b_d \in L^*.$$

Thus k is well defined only modulo L^*, and the character depends only on the class of k in $\mathbb{R}^d/L^*$. This proves $\widehat{L} \cong \mathbb{R}^d/L^*$. □

In terms of translations, we write the same character as $\chi_k(t_R) = e^{-ik\cdot R}$. The sign convention is chosen so that the translation operator acts on wavefunctions by

$$(t_R\psi)(x) = \psi(x - R).$$

Then the plane wave $\psi_k(x) = e^{ik\cdot x}$ satisfies

$$(t_R\psi_k)(x) = \psi_k(x - R) = e^{ik\cdot(x-R)} = e^{-ik\cdot R}\psi_k(x).$$

Thus $e^{ik\cdot x}$ has translation character χ_k.

Theorem 8.3 (Irreducible translation representations)**.** *Every finite dimensional irreducible unitary representation of L over $\mathbb{C}$ is one-dimensional and is given by a character*

$$R \longmapsto e^{-ik\cdot R}, \qquad k \in \mathbb{R}^d/L^*.$$

Proof. Let $\rho : L \to U(V)$ be a finite dimensional irreducible unitary representation. Since L is abelian, every operator $\rho(R)$ commutes with every $\rho(R')$. Therefore each $\rho(R)$ is an intertwiner of the representation ρ with itself. By Schur's lemma, each $\rho(R)$ must be a scalar operator:

$$\rho(R) = \chi(R)\, \mathrm{id}_V\,.$$

The map $R \mapsto \chi(R)$ is a group homomorphism from L to $U(1)$, hence a character. If $\dim V > 1$, then every one-dimensional subspace of V would be invariant, contradicting irreducibility. Therefore $\dim V = 1$, and the previous theorem gives $\chi(R) = e^{-ik\cdot R}$ for a unique $k \in \mathbb{R}^d/L^*$. □

If one forgets unitarity and studies arbitrary finite dimensional complex representations of $\mathbb{Z}^d$, irreducibility still forces the representation to be one-dimensional. Indeed, the commuting generators have a common eigenvector over $\mathbb{C}$, and the span of this eigenvector is invariant under all generators. In solid state physics the translation operators are unitary, so the relevant eigenvalues lie in $U(1)$ and are exactly the Pontryagin characters above.

Definition 8.2. *The Brillouin zone, in the representation-theoretic sense, is the compact abelian group $\widehat{L} \cong \mathbb{R}^d/L^*$. A point of this torus is called a crystal momentum or quasimomentum.*

This definition is slightly more intrinsic than the usual picture in physics. The quotient $\mathbb{R}^d/L^*$ is a torus, not a particular subset of $\mathbb{R}^d$. In computations one chooses a fundamental domain for the action of L^* on $\mathbb{R}^d$ and also calls that domain a Brillouin zone. A common choice is the first Brillouin zone, the Wigner–Seitz cell of the reciprocal lattice:

$$B_1 = \{k \in \mathbb{R}^d : |k| \leq |k - K| \text{ for all } K \in L^*\}.$$

The interior of B_1 gives one representative for each generic class in $\mathbb{R}^d/L^*$, while boundary points must be identified according to the reciprocal lattice. Thus k and $k + K$, with $K \in L^*$, label the same translation character:

$$e^{-i(k+K)\cdot R} = e^{-ik\cdot R}e^{-iK\cdot R} = e^{-ik\cdot R}.$$

Exercise 8.1. *Show directly that $\chi_k = \chi_{k'}$ as characters of L if and only if $k - k' \in L^*$.*

The important point is that crystal momentum is not introduced by imposing periodic boundary conditions. It is the Pontryagin-dual variable of the infinite translation group $L \cong \mathbb{Z}^d$. Periodic boundary conditions give a finite approximation to this picture, but they are not its origin.

Finite periodic approximations. If one replaces L by the finite quotient $L/NL \cong (\mathbb{Z}/N\mathbb{Z})^d$, then the dual group becomes a finite subgroup of the Brillouin torus:

$$\widehat{L/NL} \cong \frac{(1/N)L^*}{L^*}.$$

In coordinates, the allowed momenta are

$$k = \frac{m_1}{N} b_1 + \cdots + \frac{m_d}{N} b_d \mod L^*, \qquad m_j \in \{0, 1, \ldots, N-1\}.$$

Thus periodic boundary conditions replace the Brillouin torus by a finite grid. As $N \to \infty$, this grid becomes dense in $\mathbb{R}^d/L^*$.

Definition 8.3. *Let W be the finite dimensional internal Hilbert space inside one unit cell. For example, W may contain orbital, sublattice, and spin degrees of freedom. A tight-binding Hilbert space for an infinite crystal has the form*

$$\mathcal{H} = \ell^2(L) \otimes W.$$

Equivalently, an element of $\mathcal{H}$ is a square-summable W-valued function

$$\psi : L \to W, \qquad \sum_{R \in L} \|\psi(R)\|^2 < \infty.$$

The translation action is $t_S(|R\rangle \otimes w) = |R + S\rangle \otimes w$. In function notation this is $(t_S\psi)(R) = \psi(R - S)$.

Theorem 8.4 (Bloch–Fourier decomposition)**.** *Let $d\mu(k)$ be the normalized Haar measure on the compact torus $\widehat{L} \cong \mathbb{R}^d/L^*$. For finitely supported $\psi : L \to W$, define*

$$(\mathcal{F}\psi)(k) = \sum_{R \in L} e^{-ik \cdot R} \psi(R).$$

Then $\mathcal{F}$ extends uniquely to a unitary operator

$$\mathcal{F} : \ell^2(L) \otimes W \longrightarrow L^2(\widehat{L}, d\mu) \otimes W.$$

Under this transform, translations are diagonal: $(\mathcal{F} t_S \mathcal{F}^{-1} f)(k) = e^{-ik \cdot S} f(k)$. Equivalently,

$$\mathcal{H} \cong \int_{\widehat{L}}^{\oplus} \mathcal{H}_k \, d\mu(k), \qquad \mathcal{H}_k \cong W.$$

This is the Bloch–Fourier decomposition of the tight-binding Hilbert space. It is the Plancherel theorem for the discrete abelian group L, tensored with the finite dimensional space W [16, 11].

Proof sketch. For finitely supported functions, compute

$$(\mathcal{F} t_S \psi)(k) = \sum_{R \in L} e^{-ik \cdot R} \psi(R - S).$$

Set $R' = R - S$. Then

$$(\mathcal{F} t_S \psi)(k) = \sum_{R' \in L} e^{-ik \cdot (R' + S)} \psi(R') = e^{-ik \cdot S} (\mathcal{F}\psi)(k).$$

The unitarity follows from the character orthogonality relation

$$\int_{\widehat{L}} e^{ik \cdot (R - R')} \, d\mu(k) = \delta_{R, R'}.$$

Indeed,

$$\int_{\widehat{L}} \|(\mathcal{F}\psi)(k)\|^2 \, d\mu(k) = \sum_{R \in L} \|\psi(R)\|^2$$

for finitely supported ψ, and completion gives the result for all $\psi \in \ell^2(L) \otimes W$. □

The direct-integral notation means that a general state is a square-integrable family of fiber vectors:

$$k \longmapsto f(k) \in W.$$

A state with a perfectly sharp value of k is usually not an element of $\ell^2(L) \otimes W$. Rather, it is a generalized eigenstate of the translation operators. Formally, for $w \in W$ one writes

$$|k, w\rangle = \sum_{R \in L} e^{ik \cdot R} |R\rangle \otimes w.$$

Then $t_S|k, w\rangle = e^{-ik \cdot S}|k, w\rangle$. The sum is not square-summable, so $|k, w\rangle$ is analogous to a plane wave in ordinary Fourier analysis. Normalizable states are wave packets, namely L^2-families over the Brillouin torus.

Theorem 8.5 (Translation-invariant tight-binding Hamiltonians)**.** *Let H be a bounded operator on $\mathcal{H} = \ell^2(L) \otimes W$ such that $Ht_R = t_R H$, for all $R \in L$. Then, after the Bloch–Fourier transform, H acts fiberwise:*

$$\mathcal{F} H \mathcal{F}^{-1} = \int_{\widehat{L}}^{\oplus} H(k)\, d\mu(k),$$

where $H(k)$ is an essentially bounded measurable family of operators on W.
More concretely, suppose H is given by finite-range, or absolutely summable, hopping matrices $A_S \in \mathrm{End}(W)$:

$$H(|R\rangle \otimes w) = \sum_{S \in L} |R+S\rangle \otimes A_S w.$$

Then $H(k) = \sum_{S \in L} e^{-ik\cdot S} A_S$. If H is self-adjoint, then $A_{-S} = A_S^*$ and therefore each $H(k)$ is Hermitian.

Proof sketch. Translation invariance implies that the hopping matrix from cell R to cell $R+S$ is independent of R. Therefore the operator is a convolution operator on the lattice, with matrix-valued convolution coefficients A_S. Applying the Fourier transform converts convolution into multiplication:

$$\mathcal{F}\left(\sum_{S \in L} A_S \psi(\,\cdot - S\,)\right)(k) = \left(\sum_{S \in L} e^{-ik\cdot S} A_S\right)(\mathcal{F}\psi)(k).$$

Thus the infinite-dimensional operator H decomposes into a family of finite dimensional operators $H(k)$, one for each k in the Brillouin zone. □
The eigenvalues of $H(k)$ are the energy bands: $E_1(k), \ldots, E_{\dim W}(k)$, counted with multiplicity. Thus the representation-theoretic decomposition of the translation group is the reason that an infinite periodic Hamiltonian is studied through finite dimensional matrices parametrized by k.
For continuum Schrödinger operators, such as

$$H = -\Delta + V(x), \qquad V(x+R) = V(x) \text{ for all } R \in L,$$

the same idea leads to the Bloch–Floquet decomposition. The fibers are now infinite dimensional spaces of functions on one unit cell with k-quasiperiodic boundary conditions. Generalized eigenfunctions have the Bloch form

$$\psi_{n,k}(x) = e^{ik\cdot x} u_{n,k}(x), \qquad u_{n,k}(x+R) = u_{n,k}(x).$$

A fully rigorous treatment of this analytic decomposition is usually taken as part of Bloch–Floquet theory; see the classical work of Floquet and Bloch, and modern mathematical treatments such as Kuchment's monograph [10, 5, 13].

Example 8.1 (One-dimensional tight-binding chain)**.** *Let $L = a\mathbb{Z}, W = \mathbb{C}$. Then $L^* = \frac{2\pi}{a}\mathbb{Z}$, and the Brillouin zone is the circle*

$$\widehat{L} \cong \mathbb{R}/\frac{2\pi}{a}\mathbb{Z}.$$

A common fundamental domain is $-\frac{\pi}{a} < k \leq \frac{\pi}{a}$. Let $|n\rangle$ denote the basis vector at the lattice point na. Consider the nearest-neighbor Hamiltonian

$$H = \epsilon \sum_{n \in \mathbb{Z}} |n\rangle\langle n| - t \sum_{n \in \mathbb{Z}} \left(|n+1\rangle\langle n| + |n\rangle\langle n+1|\right).$$

The nonzero hopping coefficients are $A_0 = \epsilon, A_a = -t, A_{-a} = -t$. Therefore

$$H(k) = \epsilon - te^{-ika} - te^{ika} = \epsilon - 2t\cos(ka).$$

Since $W = \mathbb{C}$, each fiber Hamiltonian is multiplication by a scalar, and the single band is

$$E(k) = \epsilon - 2t\cos(ka).$$

The equality $E\left(k + \frac{2\pi}{a}\right) = E(k)$ reflects the fact that k is defined only modulo the reciprocal lattice.

Where finite groups re-enter. Although the translation group is infinite, the remaining symmetry at a fixed crystal momentum is finite. Let P be a point group preserving the lattice:$gL = L (g \in P)$. Then g also preserves the reciprocal lattice: $gL^* = L^*$. Hence P acts on the Brillouin torus by $g \cdot [k] = [gk]$. The stabilizer of k is

$$P_k = \{g \in P : gk \equiv k \pmod{L^*}\}.$$

This finite group is the little co-group at k. Thus the infinite translation group first decomposes the Hilbert space into fibers labelled by k, and then finite group representation theory applies inside each high-symmetry fiber.

Exercise 8.2. *Let P be a point group preserving L. Show that $gL^* = L^*$ for every $g \in P$.*

Exercise 8.3. *For $L = \mathbb{Z}a_1 \oplus \cdots \oplus \mathbb{Z}a_d$, compute explicitly the dual of the finite quotient L/NL, and show that it is*

$$\frac{(1/N)L^*}{L^*}.$$

Exercise 8.4. *For the one-dimensional chain above, impose periodic boundary conditions with N unit cells. Show that the allowed momenta are*

$$k_m = \frac{2\pi m}{Na}, \qquad m = 0, 1, \ldots, N-1,$$

modulo $2\pi/a$, and that the finite-dimensional Hamiltonian diagonalizes as

$$E(k_m) = \epsilon - 2t\cos(k_m a).$$

8.2 Space group and irreducible Brillouin zone

Translations are only part of the symmetry of a crystal. A full crystal symmetry may also contain rotations, reflections, inversions, screw rotations, and glide reflections. The natural group containing all of these operations is the Euclidean group. We write its elements in Seitz notation.

Definition 8.4. *An element of the Euclidean group is written as $\{p|\tau\}$, where $p \in O(d)$ and $\tau \in \mathbb{R}^d$. It acts on real space by $x \longmapsto px + \tau$. The multiplication rule is*

$$\{p|\tau\}\{q|\mu\} = \{pq|\tau + p\mu\}.$$

The inverse is

$$\{p|\tau\}^{-1} = \{p^{-1}| - p^{-1}\tau\}.$$

Proof. The product formula follows by applying the two affine transformations successively:

$$x \mapsto qx + \mu \mapsto p(qx + \mu) + \tau = pqx + (\tau + p\mu).$$

Thus

$$\{p|\tau\}\{q|\mu\} = \{pq|\tau + p\mu\}.$$

For the inverse, we need an element $\{r|\nu\}$ such that

$$\{p|\tau\}\{r|\nu\} = \{1|0\}.$$

The product rule gives

$$\{p|\tau\}\{r|\nu\} = \{pr|\tau + p\nu\}.$$

Thus $r = p^{-1}$ and $\tau + p\nu = 0$, so $\nu = -p^{-1}\tau$. Therefore

$$\{p|\tau\}^{-1} = \{p^{-1}| - p^{-1}\tau\}.$$

□

The case $p = 1$ gives a pure translation:

$$\{1|R\} : x \mapsto x + R.$$

As before, we write $t_R = \{1|R\}$. Then $t_R t_{R'} = t_{R+R'}$.

Definition 8.5. *A space group $\mathcal{G}$ is a discrete subgroup of the Euclidean group whose subgroup of pure translations*

$$T = \mathcal{G} \cap \{\{1|R\} : R \in \mathbb{R}^d\}$$

is a Bravais lattice. We identify T with the lattice L by writing

$$\{1|R\} = t_R, \qquad R \in L.$$

Thus $T \cong L \cong \mathbb{Z}^d$.

The discreteness condition expresses the fact that the crystal has no continuous translational or rotational symmetries. The assumption that T is a Bravais lattice says that sufficiently far translations are generated by d primitive lattice vectors. In physical dimensions $d = 2, 3$, this is the usual crystallographic setting.

Proposition 8.1. *The translation subgroup T is a normal subgroup of $\mathcal{G}$. More precisely, if $g = \{p|\tau\} \in \mathcal{G}$ and $R \in L$, then $gt_Rg^{-1} = t_{pR}$. In particular, $pL = L$.*

Proof. Using the multiplication rule and the inverse formula, we compute

$$\{p|\tau\}\{1|R\}\{p|\tau\}^{-1} = \{p|\tau\}\{1|R\}\{p^{-1}| - p^{-1}\tau\}.$$

First, $\{p|\tau\}\{1|R\} = \{p|\tau + pR\}$. Then

$$\{p|\tau + pR\}\{p^{-1}| - p^{-1}\tau\} = \{1|\tau + pR + p(-p^{-1}\tau)\} = \{1|pR\}.$$

Thus $gt_Rg^{-1} = t_{pR}$. Since $g, t_R, g^{-1} \in \mathcal{G}$, the element t_{pR} also belongs to $\mathcal{G}$. Hence $pR \in L$. Applying the same argument to g^{-1} gives $p^{-1}L \subset L$, or equivalently $L \subset pL$. Therefore $pL = L$. □

Definition 8.6. *The point homomorphism is the map*

$$\pi : \mathcal{G} \longrightarrow O(d), \qquad \pi(\{p|\tau\}) = p.$$

The point group of $\mathcal{G}$ is $P = \pi(\mathcal{G}) \subset O(d)$.

The kernel of π consists exactly of the pure translations in $\mathcal{G}$: $\ker \pi = T$. Therefore the point group is the quotient $P \cong \mathcal{G}/T$. This quotient forgets where the symmetry operation translates the point and remembers only its orthogonal part.

Theorem 8.6 (Finiteness of the point group)**.** *The point group P of a space group is finite.*

Proof. By the previous proposition, every $p \in P$ satisfies $pL = L$. Choose a lattice basis $L = \mathbb{Z}a_1 \oplus \cdots \oplus \mathbb{Z}a_d$. With respect to this basis, p is represented by a matrix $A_p \in GL(d, \mathbb{Z})$. Indeed, $pL = L$ means that each pa_j is an integral linear combination of the basis vectors $a_1, \ldots, a_d$, and invertibility over $\mathbb{Z}$ follows from $p^{-1}L = L$. Let $G = (a_i \cdot a_j)_{i,j}$ be the Gram matrix of the lattice basis. Since $p \in O(d)$, it preserves the Euclidean inner product. Therefore

$$(pa_i) \cdot (pa_j) = a_i \cdot a_j.$$

In matrix form this says $A_p^T G A_p = G$. Because G is positive definite, the columns of A_p have bounded G-length. But each column is an integer vector. There are only finitely many integer vectors with bounded G-length. Hence there are only finitely many possible matrices A_p, and therefore only finitely many possible elements $p \in P$. □

Thus every space group fits into a short exact sequence

$$1 \longrightarrow T \longrightarrow \mathcal{G} \xrightarrow{\pi} P \longrightarrow 1,$$

where $T \cong \mathbb{Z}^d$ is infinite and abelian, while P is finite. This exact sequence is one of the main reasons why space-group representation theory combines Fourier analysis for translations with finite-group representation theory for point symmetries.

Definition 8.7. *A space group is called symmorphic if the exact sequence*

$$1 \longrightarrow T \longrightarrow \mathcal{G} \longrightarrow P \longrightarrow 1$$

splits. Equivalently, $\mathcal{G}$ is symmorphic if there exists a group homomorphism $s : P \longrightarrow \mathcal{G}$ such that $\pi \circ s = \mathrm{id}_P$. If no such splitting exists, the space group is called nonsymmorphic.

Proposition 8.2. *If $\mathcal{G}$ is symmorphic, then $\mathcal{G} \cong L \rtimes P$. In this case every element of $\mathcal{G}$ can be written uniquely as $t_R s(p), R \in L,\ p \in P$, and the multiplication law is $(R, p)(R', p') = (R + pR', pp')$.*

Proof. Suppose the exact sequence splits, and let $s : P \to \mathcal{G}$ be a splitting. If $g \in \mathcal{G}$ and $\pi(g) = p$, then $\pi(g\, s(p)^{-1}) = 1$. Hence $g\, s(p)^{-1} \in T$. Therefore there is some $R \in L$ such that $g = t_R s(p)$. This expression is unique because if $t_R s(p) = t_{R'} s(p')$, then applying π gives $p = p'$, and hence $t_R = t_{R'}$. The product of two such elements is

$$t_R s(p) t_{R'} s(p') = t_R\big(s(p) t_{R'} s(p)^{-1}\big) s(p) s(p').$$

Since $s(p)$ has point part p, the conjugation formula gives $s(p)t_{R'}s(p)^{-1} = t_{pR'}$. Also, s is a homomorphism, so $s(p)s(p') = s(pp')$. Thus

$$t_R s(p) t_{R'} s(p') = t_R t_{pR'} s(pp') = t_{R+pR'} s(pp').$$

This is exactly the semidirect product law $(R, p)(R', p') = (R + pR', pp')$. □

In a nonsymmorphic space group, some point operations can only be realized together with fractional translations. The standard examples are screw rotations and glide reflections. A screw rotation is a rotation followed by a fractional translation along the rotation axis. A glide reflection is a reflection followed by a fractional translation parallel to the mirror plane. These fractional translations cannot all be removed by changing the origin.
The classical structural results for crystallographic groups are the Bieberbach theorems. We will not use their full strength here, but they justify the general viewpoint that a crystallographic group is an extension of a lattice translation group by a finite point group [2, 3].

Definition 8.8. *The point group acts on reciprocal space by* $k \longmapsto pk$*, where* pk *is defined by the pairing rule*

$$(pk) \cdot R = k \cdot p^{-1}R \qquad (R \in L).$$

Since $p \in O(d)$*, this is the usual orthogonal action on vectors, but the pairing formula is the most invariant way to write it.*

Proposition 8.3. *If* $pL = L$*, then* $pL^* = L^*$*. Consequently, the point group acts on the Brillouin torus* $\mathbb{R}^d/L^*$*.*

Proof. Let $K \in L^*$. We must show that $pK \in L^*$. For any $R \in L$, we have $p^{-1}R \in L$ because $pL = L$. Therefore

$$e^{i(pK)\cdot R} = e^{iK\cdot p^{-1}R} = 1.$$

Thus $pK \in L^*$. Hence $pL^* \subset L^*$. Applying the same argument to p^{-1} gives $p^{-1}L^* \subset L^*$, which is equivalent to $L^* \subset pL^*$. Therefore $pL^* = L^*$. It follows that if k and $k + K$ represent the same point of $\mathbb{R}^d/L^*$, then

$$pk \quad \text{and} \quad p(k + K) = pk + pK$$

also represent the same point. Hence the action descends to the quotient torus. □

Thus the point group does not merely act on real space; it also acts on the Brillouin zone. This action is the reason why band diagrams have symmetry-related momenta and why one can reduce computations to an irreducible Brillouin zone.

Proposition 8.4. *Let* $g = \{p|\tau\} \in \mathcal{G}$*. Then* $gt_Rg^{-1} = t_{pR}$*. Consequently, if a generalized Bloch state has translation character* $\chi_k(R) = e^{-ik\cdot R}$*, then* g *sends it to a generalized Bloch state with translation character* $\chi_{pk}(R) = e^{-i(pk)\cdot R}$*.*

Proof. The conjugation identity $gt_Rg^{-1} = t_{pR}$ was proved above. Let ψ be a generalized Bloch state with character χ_k, so that $t_R\psi = e^{-ik\cdot R}\psi$. Then $t_R(g\psi) = g(g^{-1}t_Rg)\psi$. Since $g^{-1}t_Rg = t_{p^{-1}R}$, we obtain

$$t_R(g\psi) = g\, t_{p^{-1}R}\psi = e^{-ik\cdot p^{-1}R}g\psi.$$

By definition of the point-group action on k,

$$k \cdot p^{-1}R = (pk) \cdot R.$$

Therefore $t_R(g\psi) = e^{-i(pk)\cdot R}g\psi$. Thus $g\psi$ has translation character χ_{pk}. □

In operator language, if $\mathcal{H}_k$ denotes the Bloch fiber at crystal momentum k, then an element $g = \{p|\tau\}$ maps $\mathcal{H}_k \longrightarrow \mathcal{H}_{pk}$. The translational part τ does not change the momentum label. It contributes a phase and may act nontrivially on internal degrees of freedom, but the momentum label is transformed only by the point part p.
For scalar continuum wavefunctions, the action is especially transparent. Let

$$(U_g\psi)(x) = \psi(g^{-1}x) = \psi(p^{-1}(x - \tau)).$$

If $\psi_k(x) = e^{ik\cdot x}u_k(x), u_k(x + R) = u_k(x)$, then

$$(U_g\psi_k)(x) = e^{ik\cdot p^{-1}(x-\tau)}u_k(p^{-1}(x - \tau)).$$

Since $k \cdot p^{-1}x = (pk) \cdot x$, this becomes

$$(U_g\psi_k)(x) = e^{-i(pk)\cdot\tau}e^{i(pk)\cdot x}u_k(p^{-1}(x - \tau)).$$

Thus $U_g\psi_k$ is a Bloch function at momentum pk, and the explicit phase coming from the translational part of g is $e^{-i(pk)\cdot\tau}$.

Definition 8.9. *The star of k is the point-group orbit*

$$Pk = \{pk \mod L^* : p \in P\}.$$

The little co-group of k is the stabilizer of k in the point group action:

$$P_k = \{p \in P : pk = k \mod L^*\}.$$

The little group of k inside the space group is

$$\mathcal{G}_k = \{\{p|\tau\} \in \mathcal{G} : pk = k \mod L^*\}.$$

The star describes all momenta related to k by point-group symmetry. The little co-group describes the point symmetries that keep k fixed modulo a reciprocal lattice vector. The little group remembers the full space-group elements above those point symmetries, including possible fractional translations.

Proposition 8.5. *The little group fits into an exact sequence*

$$1 \longrightarrow T \longrightarrow \mathcal{G}_k \longrightarrow P_k \longrightarrow 1.$$

In particular, $\mathcal{G}_k/T \cong P_k$.

Proof. The map $\pi : \mathcal{G} \to P$ restricts to a map $\pi : \mathcal{G}_k \to P_k$ because if $g = \{p|\tau\} \in \mathcal{G}_k$, then by definition $pk = k \mod L^*$. The kernel of this restricted map consists of elements of $\mathcal{G}_k$ with point part 1. These are exactly the translations T. Finally, if $p \in P_k$, then by definition of $P = \pi(\mathcal{G})$ there is some $\{p|\tau\} \in \mathcal{G}$. Since $p \in P_k$, this element lies in $\mathcal{G}_k$. Hence the restricted map is onto P_k. Therefore $1 \to T \to \mathcal{G}_k \to P_k \to 1$ is exact. □

A generic point k in the Brillouin zone usually has trivial little co-group. Special points, lines, and planes can have nontrivial little co-groups. These are the high-symmetry points, high-symmetry lines, and high-symmetry planes familiar from band-structure plots.
If $g \in \mathcal{G}_k$, then g preserves the Bloch fiber $\mathcal{H}_k$. Therefore states at k can be decomposed into irreducible representations of the little group $\mathcal{G}_k$. Since the translation subgroup T acts on $\mathcal{H}_k$ by the scalar character $t_R \mapsto e^{-ik\cdot R}$, one often describes the relevant labels as irreducible representations of the little co-group P_k, possibly with additional phase factors coming from fractional translations. These are often called small representations.

Definition 8.10. *A small representation at k is a representation*

$$\rho : \mathcal{G}_k \to GL(V)$$

such that every translation acts by the Bloch character:

$$\rho(t_R) = e^{-ik\cdot R}\,\mathrm{id}_V \qquad (R \in L).$$

The condition above means that the representation has already fixed its translation character. The remaining nontrivial information is how the little group acts inside the Bloch fiber.

Theorem 8.7 (Little-group method for space groups)**.** *Irreducible space-group representations with translation character in the star of k are obtained by inducing small representations of $\mathcal{G}_k$ to the full space group $\mathcal{G}$. Conversely, the irreducible representations relevant to a given star arise in this way, up to the usual equivalence relations. This is the space-group version of Wigner's little-group method and Mackey's theory of induced representations [18, 15, 6].*

We will use this theorem as a structural guide rather than prove it here. Its content is exactly what the physics picture suggests: first fix a translation character k, then classify the remaining symmetry action using the little group at k, and finally generate the whole star by applying the space group.

Definition 8.11. *An irreducible Brillouin zone is a choice of representatives for the orbits of the point group P acting on $\mathbb{R}^d/L^*$. Equivalently, it is a fundamental domain for the action of P on the Brillouin torus.*

The irreducible Brillouin zone is not unique. Different choices of fundamental domain give different-looking regions but the same quotient information. Boundary points must be treated carefully because they may be identified by reciprocal lattice vectors or by point-group operations. These boundary strata are precisely where little co-groups tend to become larger.

Proposition 8.6. *Let H be a Hamiltonian commuting with the full space-group action: $U_g H U_g^{-1} = H, (g \in \mathcal{G})$. Then the Bloch Hamiltonians satisfy $H(pk) \cong H(k), (p \in P)$. Consequently, the energy bands obey $E_n(pk) = E_n(k)$, with the usual convention that band labels may be permuted at degeneracies.*

Proof. Let $g = \{p|\tau\} \in \mathcal{G}$. The operator U_g maps the Bloch fiber $\mathcal{H}_k$ to the Bloch fiber $\mathcal{H}_{pk}$. Since H commutes with U_g, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{H}_k & \xrightarrow{H(k)} & \mathcal{H}_k \\ \downarrow U_g & & \downarrow U_g \\ \mathcal{H}_{pk} & \xrightarrow{H(pk)} & \mathcal{H}_{pk}. \end{array}$$

Therefore $H(pk)U_g = U_g H(k)$, or equivalently $H(pk) = U_g H(k) U_g^{-1}$ as operators on the corresponding fibers. Thus $H(pk)$ and $H(k)$ are unitarily equivalent. Their spectra are equal, so $E_n(pk) = E_n(k)$ after ordering the eigenvalues consistently. □

This proposition explains why one only needs to compute the band structure on an irreducible Brillouin zone. The band structure on the rest of the Brillouin zone is generated by applying the point group.

Example 8.2 (Square lattice)**.** *Let $L = a\mathbb{Z}e_x \oplus a\mathbb{Z}e_y$. The reciprocal lattice is*

$$L^* = \frac{2\pi}{a}\mathbb{Z}e_x \oplus \frac{2\pi}{a}\mathbb{Z}e_y.$$

A convenient first Brillouin zone is the square

$$-\frac{\pi}{a} \le k_x \le \frac{\pi}{a}, -\frac{\pi}{a} \le k_y \le \frac{\pi}{a}.$$

The point group of the square lattice is the dihedral group D_4. It is generated by a fourfold rotation $C_4 : (k_x, k_y) \mapsto (-k_y, k_x)$ and a reflection, for example $m_x : (k_x, k_y) \mapsto (k_x, -k_y)$. A convenient irreducible Brillouin zone is the triangular region $0 \le k_y \le k_x \le \frac{\pi}{a}$.
The standard high-symmetry points are

$$\Gamma = (0,0), \qquad X = \left(\frac{\pi}{a}, 0\right), \qquad M = \left(\frac{\pi}{a}, \frac{\pi}{a}\right).$$

At Γ, every point-group operation fixes k, so $P_\Gamma = D_4$. At M, every point-group operation also fixes M modulo reciprocal lattice vectors, so $P_M = D_4$. At X, the orbit contains both

$$\left(\frac{\pi}{a}, 0\right) \quad \text{and} \quad \left(0, \frac{\pi}{a}\right),$$

so the stabilizer is smaller. In this case $P_X \cong D_2$.
Thus Bloch states at Γ and M are labeled by irreducible representations of D_4, while Bloch states at X are labeled by irreducible representations of the smaller little co-group D_2. Along a generic point of the irreducible Brillouin zone, the little co-group is trivial. Along high-symmetry lines, the little co-group is usually an order-two reflection group. This is the same little-group organization used in crystallographic tables of space-group irreducible representations [6, 1].

For nonsymmorphic space groups, one must keep track of the translational parts of space-group elements. These translational parts can produce phase factors in Bloch representations, especially at the boundary of the Brillouin zone.

Definition 8.12. *Let $p, q \in P_k$. Choose representatives*

$$s(p) = \{p|\tau_p\}, \qquad s(q) = \{q|\tau_q\}, \qquad s(pq) = \{pq|\tau_{pq}\}$$

inside the little group $\mathcal{G}_k$. Define $R(p,q) = \tau_p + p\tau_q - \tau_{pq}$. Then $R(p,q) \in L$ and $s(p)s(q) = t_{R(p,q)}s(pq)$.

Proof. By the multiplication rule,

$$s(p)s(q) = \{p|\tau_p\}\{q|\tau_q\} = \{pq|\tau_p + p\tau_q\}.$$

On the other hand,

$$t_{R(p,q)}s(pq) = \{1|R(p,q)\}\{pq|\tau_{pq}\} = \{pq|R(p,q) + \tau_{pq}\}.$$

With

$$R(p,q) = \tau_p + p\tau_q - \tau_{pq},$$

we get $R(p,q) + \tau_{pq} = \tau_p + p\tau_q$. Hence $s(p)s(q) = t_{R(p,q)}s(pq)$. Since both $s(p)s(q)$ and $s(pq)$ are elements of the space group with the same point part pq, their quotient is a pure translation in $\mathcal{G}$. Thus $R(p,q) \in L$. □

On the Bloch fiber $\mathcal{H}_k$, the translation $t_{R(p,q)}$ acts by the scalar $e^{-ik\cdot R(p,q)}$. Therefore the matrices assigned to the little co-group representatives may satisfy

$$U_p(k)U_q(k) = e^{-ik\cdot R(p,q)}U_{pq}(k),$$

rather than the ordinary multiplication law $U_p(k)U_q(k) = U_{pq}(k)$. This is a projective representation of the little co-group.

Proposition 8.7. *The factor $\omega_k(p,q) = e^{-ik\cdot R(p,q)}$ satisfies the cocycle identity*

$$\omega_k(p,q)\omega_k(pq,r) = \omega_k(q,r)\omega_k(p,qr)$$

for $p, q, r \in P_k$, up to the standard equivalence produced by changing the choice of representatives $s(p)$.

Proof. Associativity in $\mathcal{G}$ gives

$$(s(p)s(q))s(r) = s(p)(s(q)s(r)).$$

Using $s(p)s(q) = t_{R(p,q)}s(pq)$, the left-hand side becomes

$$t_{R(p,q)}s(pq)s(r) = t_{R(p,q)}t_{R(pq,r)}s(pqr).$$

The right-hand side becomes $s(p)t_{R(q,r)}s(qr)$. Since $s(p)t_{R(q,r)}s(p)^{-1} = t_{pR(q,r)}$, this is

$$t_{pR(q,r)}s(p)s(qr) = t_{pR(q,r)}t_{R(p,qr)}s(pqr).$$

Therefore

$$R(p,q) + R(pq,r) = pR(q,r) + R(p,qr).$$

Applying the Bloch character $R \mapsto e^{-ik\cdot R}$ gives

$$e^{-ik\cdot R(p,q)}e^{-ik\cdot R(pq,r)} = e^{-ik\cdot pR(q,r)}e^{-ik\cdot R(p,qr)}.$$

Since $p \in P_k$, we have $p^{-1}k = k \mod L^*$. Thus $e^{-ik\cdot pR(q,r)} = e^{-ik\cdot R(q,r)}$. Hence

$$\omega_k(p,q)\omega_k(pq,r) = \omega_k(q,r)\omega_k(p,qr).$$

Changing the representatives $s(p)$ changes $R(p,q)$ by a coboundary, so the projective class of ω_k is independent of the particular section. □

At the zone center $\Gamma = 0$, all translation phases are trivial: $e^{-i0\cdot R} = 1$. Thus nonsymmorphic fractional translations are invisible in this particular factor system at Γ. At the Brillouin-zone boundary, however, phases such as $e^{-ik\cdot R} = -1$ can occur. This is one of the main sources of enforced degeneracies and unusual compatibility relations in nonsymmorphic crystals.

Example 8.3 (Glide reflection)**.** *Consider a two-dimensional rectangular lattice with primitive vector ae_x in the x-direction. Let $g = \{m_y|\frac{a}{2}e_x\}$, where m_y is reflection across the x-axis: $m_y(x,y) = (x,-y)$. Then*

$$g^2 = \{m_y|\tfrac{a}{2}e_x\}\{m_y|\tfrac{a}{2}e_x\} = \{1|ae_x\} = t_{ae_x}.$$

Thus on a Bloch fiber, $U_g(k)^2 = e^{-ik_x a}$. At the Brillouin-zone boundary $k_x = \frac{\pi}{a}$, we get $U_g(k)^2 = -1$. Hence the glide eigenvalues are $\pm i$ rather than ± 1. This phase is a direct representation-theoretic effect of the fractional translation in the glide reflection.

Example 8.4 (Screw rotation)**.** *Let $g = \{C_n|\frac{1}{n}c\}$, where C_n is an n-fold rotation about an axis and c is a lattice translation along that axis. Then $g^n = t_c$. On a Bloch fiber, $U_g(k)^n = e^{-ik\cdot c}$. At momenta satisfying $k\cdot c = \pi \mod 2\pi$, the screw eigenvalues are nth roots of -1 rather than nth roots of 1. Again, the fractional translation changes the little-group representation at the Brillouin-zone boundary.*

Definition 8.13. *Let $q \in \mathbb{R}^d$ be the position of a site in the crystal. The site stabilizer of q is $\mathcal{G}_q = \{g \in \mathcal{G} : gq = q\}$. More generally, the stabilizer of the translation orbit $q + L$ is*

$$\mathcal{G}_{[q]} = \{g \in \mathcal{G} : gq = q + R \text{ for some } R \in L\}.$$

The finite point group associated with this site is obtained by passing to the appropriate quotient by translations.

The distinction is useful. The group $\mathcal{G}_q$ fixes the chosen physical site exactly, while $\mathcal{G}_{[q]}$ preserves the whole translation orbit of that site. In the construction of localized orbitals, one usually starts with a finite-dimensional representation of the site stabilizer or its finite point-group quotient.

Definition 8.14. *Let σ be a finite-dimensional representation of the site stabilizer $\mathcal{G}_q$ on a vector space V. The induced representation $\mathrm{Ind}_{\mathcal{G}_q}^{\mathcal{G}} \sigma$ is called a band representation.*

Physically, the vector space V describes the internal degrees of freedom of an orbital centered at q: for example, an s orbital, a triple of p orbitals, or spinful orbitals. Inducing from $\mathcal{G}_q$ to $\mathcal{G}$ means that one places symmetry-related copies of this orbital throughout the crystal and lets the full space group act on all of them. This is exactly the induction construction from finite group representation theory, but now applied to a subgroup of a space group.
If the orbit of q contains several sites in one unit cell, the induced representation has one copy of V for each symmetry-related site in the cell. After the Bloch–Fourier transform, this becomes a family of finite-dimensional Bloch representations over the Brillouin zone. At a high-symmetry momentum k, this family decomposes according to irreducible representations of the little group $\mathcal{G}_k$.
This point of view is the bridge between local orbital data and momentum-space band labels. Local orbitals are classified by site-symmetry representations; Bloch states are classified by little-group representations; band representations connect the two.

Exercise 8.5. *Let $g = \{p|\tau\} \in \mathcal{G}_k$. For a scalar Bloch function*

$$\psi_k(x) = e^{ik\cdot x}u_k(x), \qquad u_k(x+R) = u_k(x),$$

show directly that

$$(U_g\psi_k)(x) = e^{-i(pk)\cdot\tau}e^{i(pk)\cdot x}u_k(p^{-1}(x-\tau)).$$

Explain why the translational phase is trivial at $k = 0$ but may be nontrivial at the Brillouin-zone boundary.

Exercise 8.6. *Let P be the point group of a space group with translation lattice L. Prove that $pL^* = L^*$ for every $p \in P$. Conclude that P acts on the Brillouin torus $\mathbb{R}^d/L^*$.*

Exercise 8.7. *For the square lattice, verify explicitly that*

$$P_\Gamma \cong D_4, \qquad P_M \cong D_4, \qquad P_X \cong D_2.$$

Identify the point-group elements that stabilize X modulo reciprocal lattice vectors.

Exercise 8.8. *Let $s(p) = \{p|\tau_p\}$ be representatives for a little co-group P_k. Prove that $R(p,q) = \tau_p + p\tau_q - \tau_{pq}$ belongs to L, and compute the corresponding factor $\omega_k(p,q) = e^{-ik\cdot R(p,q)}$. Show that at $k = 0$ the factor system is trivial.*

8.3 Magnetic groups and corepresentations

Ordinary point groups and space groups describe symmetries of positions. Magnetic structures require one additional operation: time reversal. Classically, time reversal leaves positions unchanged but reverses velocities, momenta, angular momenta, spins, and magnetic moments. Quantum mechanically, time reversal is represented not by a unitary operator but by an antiunitary operator. This is the reason ordinary representation theory must be replaced by Wigner's corepresentation theory when antiunitary symmetries are present [17, 7, 9, 8].
The basic physical distinction is this. A spatial symmetry acts linearly on wavefunctions, while time reversal acts antilinearly: it complex conjugates coefficients. Since Bloch phases are complex numbers, this antilinearity has an immediate consequence in momentum space:

$$e^{-ik\cdot R} \longmapsto e^{+ik\cdot R}.$$

Thus time reversal sends crystal momentum k to $-k$, up to reciprocal lattice vectors. This simple sign change is the source of many familiar band-theoretic facts, including Kramers pairs at time-reversal-invariant momenta and antiunitary degeneracies in antiferromagnets.

Definition 8.15. *Let $1'$ be a formal operation of order two, called time reversal in the crystallographic magnetic-group setting. It satisfies*

$$(1')^2 = 1$$

and commutes with all spatial operations. If $g \in E(d)$ is a Euclidean operation, then we write

$$g' = 1'g = g1'$$

for the same spatial operation followed by time reversal.

The prime notation is crystallographic. An unprimed operation is a pure spatial operation. A primed operation is a spatial operation composed with time reversal. The symbol $1'$ should be regarded as a classical symmetry operation. Its quantum representative may square to $+1$ or to -1, depending on the Hilbert space on which it acts.

Definition 8.16. *A magnetic point group, or more generally a magnetic space group, is a subgroup*

$$M \subset E(d) \times \{1, 1'\}.$$

An element of M has the form

$$m = (\{p|\tau\}, \epsilon), \qquad p \in O(d), \quad \tau \in \mathbb{R}^d, \quad \epsilon \in \{1, 1'\}.$$

We usually suppress the pair notation and write such an element as

$$\{p|\tau\} \quad \text{or} \quad \{p|\tau\}'.$$

Since $1'$ commutes with the spatial operation, the multiplication law is the same affine multiplication law as before, together with multiplication of the prime labels:

$$\{p|\tau\}^{\epsilon}\{q|\mu\}^{\delta} = \{pq|\tau + p\mu\}^{\epsilon\delta}.$$

Here $\epsilon\delta$ is unprimed if ϵ and δ are both unprimed or both primed, and primed otherwise. Thus the product of two antiunitary, or primed, operations is unitary.

Definition 8.17. *Define the sign homomorphism $\eta : M \longrightarrow \{+1, -1\}$ by*

$$\eta(m) = \begin{cases} +1, & m \text{ is unprimed,} \\ -1, & m \text{ is primed.} \end{cases}$$

The subgroup $M_0 = \ker \eta$ is called the unitary subgroup of M.

If M contains no primed elements, then $M = M_0$ is just an ordinary point group or ordinary space group. If M contains at least one primed element, then M_0 has index two in M, and

$$M = M_0 \sqcup aM_0$$

for any fixed antiunitary element $a \in M \setminus M_0$. The second coset aM_0 consists entirely of primed operations.

Definition 8.18. *In crystallography, magnetic groups are often divided into four standard types.*

Type I: $M = G$.

Type II: $M = G \sqcup 1'G$.

Type III: $M = H \sqcup 1'(G \setminus H), \quad H \subset G$ *of index* 2.

Type IV: the antiunitary coset contains translations not in the unitary translation lattice.

Here G is an ordinary crystallographic point group or space group.

Type I groups describe systems with no time-reversal symmetry. Type II groups are often called grey groups; they contain time reversal as a separate symmetry and commonly describe paramagnets before magnetic ordering. Type III and Type IV groups are black-white, or Shubnikov, groups. They describe magnetic structures in which some spatial symmetries are lost but reappear when combined with time reversal. Type IV groups are especially important for antiferromagnets because an operation such as a half-translation followed by time reversal can be a symmetry even when the half-translation alone is not.
A magnetic moment is an axial vector. If $p \in O(3)$ is the orthogonal part of a spatial operation, then a polar vector transforms as $v \longmapsto pv$, whereas an axial vector transforms as $m \longmapsto \det(p)\, pm$. Time reversal leaves positions unchanged but reverses magnetic moments: $m \longmapsto -m$. Thus a mirror operation may fail to preserve a magnetic moment by itself, while the same mirror followed by time reversal may preserve it.

Example 8.5 (A magnetic point group from a ferromagnetic moment)**.** *Suppose a system has a magnetic moment $m = m_0\hat{z}$ and an underlying square geometry with point group C_{4v}. The rotations C_4^n preserve $\hat{z}$, so they remain ordinary unitary symmetries of the magnetic state.*
A vertical mirror has determinant -1. Since the moment is an axial vector, the mirror sends $\hat{z} \longmapsto -\hat{z}$. Therefore the mirror is not a symmetry of the ferromagnetic state by itself. However, the mirror followed by time reversal sends $\hat{z} \longmapsto -\hat{z} \longmapsto \hat{z}$. Thus the magnetic point group has the schematic form

$$M = C_4 \sqcup 1'\sigma_v C_4.$$

The unitary subgroup is C_4, while the antiunitary coset consists of the vertical mirrors multiplied by time reversal.

Definition 8.19. *Let V be a complex vector space. A map $A : V \to V$ is linear if*

$$A(cv + w) = cA(v) + A(w),$$

and antilinear if

$$A(cv + w) = c^* A(v) + A(w).$$

If V is a Hilbert space, a linear map U is unitary if

$$\langle Uv|Uw\rangle = \langle v|w\rangle.$$

An antilinear map A is antiunitary if

$$\langle Av|Aw\rangle = \langle w|v\rangle.$$

The last formula uses the physicists' convention that the inner product is linear in the ket and conjugate-linear in the bra. Antiunitarity is not merely unitarity plus complex conjugation as an afterthought; it changes how scalars pass through the operator. In particular, $A(cv) = c^* A(v)$.

Proposition 8.8. *The product of two linear maps is linear. The product of a linear map and an antilinear map is antilinear. The product of two antilinear maps is linear.*

Proof. Let A and B be antilinear. Then

$$(AB)(cv) = A(c^* Bv) = cA(Bv),$$

so AB is linear. If A is linear and B is antilinear, then

$$(AB)(cv) = A(c^* Bv) = c^* A(Bv),$$

so AB is antilinear. The other cases are immediate. □

Definition 8.20. *A strict corepresentation of a magnetic group M on a complex vector space V assigns to every $m \in M$ an operator $D(m)$ such that $D(mn) = D(m)D(n)$. If $\eta(m) = +1$, then $D(m)$ is linear. If $\eta(m) = -1$, then $D(m)$ is antilinear. A unitary corepresentation is one in which the linear operators are unitary and the antilinear operators are antiunitary.*

For spinless systems this strict definition is often sufficient. For spinful electrons one usually needs a double or projective version. The reason is that a 2π rotation is the identity operation in the classical point group, but it acts as -1 on a spin-$\frac{1}{2}$ wavefunction. Similarly, the classical operation $1'$ satisfies $(1')^2 = 1$, while the quantum time-reversal operator for a single spin-$\frac{1}{2}$ satisfies $\Theta^2 = -1$. Thus spinful electrons are naturally described either by double magnetic groups or by projective corepresentations.

Definition 8.21. *A projective corepresentation allows multiplication up to phase:*

$$D(m)D(n) = \omega(m, n)D(mn),$$

where $\omega(m, n) \in U(1)$. The function ω is called a factor system. For spinful time reversal, one may encode $\Theta^2 = -1$ by taking

$$\omega(1', 1') = -1.$$

Equivalently, one may enlarge the group to a double magnetic group in which the central element corresponding to a 2π rotation is represented by $-I$.

In physics it is common to say simply "corepresentation" even when a double-valued or projective corepresentation is meant. In what follows, the distinction will be stated explicitly when the sign of a square, such as $\Theta^2 = \pm 1$, matters.

Example 8.6 (The two smallest time-reversal corepresentations)**.** *Let $M = \{e, 1'\}$. For a spinless one-dimensional Hilbert space, take $D(e) = 1, D(1') = K$, where K is complex conjugation. Then $D(1')^2 = K^2 = 1$. There is no forced twofold degeneracy.*
For a spin-$\frac{1}{2}$ Kramers pair, take $V = \mathbb{C}^2$ and $D(1') = i\sigma_y K$. Since

$$i\sigma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is real, we get

$$(i\sigma_y K)^2 = (i\sigma_y)(i\sigma_y)^* = (i\sigma_y)^2 = -I.$$

This two-dimensional projective corepresentation is the basic algebraic source of Kramers degeneracy.

Proposition 8.9 (Time reversal for spin s)**.** *In the standard S_z-basis, the time-reversal operator for a spin s degree of freedom may be written as $\Theta_s = e^{-i\pi S_y}K$. It satisfies $\Theta_s^2 = (-1)^{2s}$. Thus $\Theta_s^2 = +1$ for integer spin and $\Theta_s^2 = -1$ for half-integer spin.*

Proof. In the S_z-basis, the matrix S_y is purely imaginary, so $S_y^* = -S_y$. Therefore

$$\left(e^{-i\pi S_y}\right)^* = e^{-i\pi S_y}.$$

Hence

$$\Theta_s^2 = e^{-i\pi S_y}Ke^{-i\pi S_y}K = e^{-i\pi S_y}\left(e^{-i\pi S_y}\right)^* = e^{-i2\pi S_y}.$$

A 2π rotation acts on spin s as multiplication by $(-1)^{2s}$. Therefore $\Theta_s^2 = (-1)^{2s}$. □

In electronic band theory, the most common case is spin $\frac{1}{2}$, so one usually writes

$$\Theta = i\sigma_y K, \qquad \Theta^2 = -1.$$

If spin-orbit coupling is present, the spin and orbital motion cannot be separated, but the same antiunitary operator acts on the full internal Hilbert space.

Theorem 8.8 (Wigner corepresentation classification)**.** *Let*

$$M = M_0 \sqcup aM_0$$

be a finite magnetic group with unitary subgroup M_0 and one fixed antiunitary element a. Let ρ be an irreducible unitary representation of M_0. Define the conjugate-twisted representation

$$\rho^a(h) = \rho(a^{-1}ha)^* \qquad (h \in M_0).$$

Then the irreducible corepresentations of M are organized by comparing ρ with ρ^a. There are three cases:

$$\begin{aligned} &\textit{Case (a):} && \rho^a \simeq \rho \textit{ and the antiunitary extension has square } +1. \\ &\textit{Case (b):} && \rho^a \simeq \rho \textit{ and the antiunitary extension has square } -1. \\ &\textit{Case (c):} && \rho^a \not\simeq \rho. \end{aligned}$$

In case (a), the corepresentation has the same dimension as ρ. In case (b), one obtains a Kramers-type doubling. In case (c), the antiunitary element pairs two inequivalent unitary representations, and the corepresentation has dimension $2\dim\rho$.

Equivalently, one may use Herring's criterion. Let χ_ρ be the character of ρ. Since $B^2 \in M_0$ for every antiunitary element $B \in M \setminus M_0$, the sum

$$\mathcal{H}(\rho) = \sum_{B \in M \setminus M_0} \chi_\rho(B^2)$$

is defined using only the ordinary character of M_0. The three cases are distinguished by

$$\mathcal{H}(\rho) = \begin{cases} +|M_0|, & \textit{case (a)}, \\ -|M_0|, & \textit{case (b)}, \\ 0, & \textit{case (c)}. \end{cases}$$

For projective or double-valued representations, the same idea holds with the appropriate factor system included. This classification is a standard result of Wigner's corepresentation theory and its application to magnetic groups [17, 7, 8].

For band theory, the most important lesson of Wigner's theorem is physical rather than formal. Antiunitary symmetries can do two different things. They may leave an ordinary irreducible representation invariant, in which case one must ask whether the antiunitary square is $+1$ or -1. Or they may send one representation to a distinct conjugate partner. The second possibility is why antiunitary symmetries often relate states at different momenta, such as k and $-k$.

We now pass from finite magnetic point groups to magnetic space groups. For a magnetic space group, the translation group relevant for Bloch decomposition is the unitary translation subgroup. We write

$$T_0 = M_0 \cap \{\text{pure translations}\} = \{t_R : R \in L\},$$

where L is the magnetic translation lattice. In a ferromagnet this may be the original crystallographic lattice. In an antiferromagnet it may be a larger real-space unit cell, hence a smaller Brillouin zone.

Definition 8.22. *Let M be a magnetic space group with unitary translation lattice L. If*

$$m = \{p_m|\tau_m\} \quad or \quad m = \{p_m|\tau_m\}'$$

is an element of M, define its action on crystal momentum by

$$k \longmapsto \eta(m)p_m k.$$

The magnetic little group of k is

$$M_k = \{m \in M : \eta(m)p_m k = k \mod L^*\}.$$

The extra sign for primed elements is the momentum-space signature of complex conjugation. A unitary spatial operation sends $k \longmapsto p_m k$. An antiunitary operation sends $k \longmapsto -p_m k$. This is one of the most important formulas in magnetic band theory.

Proposition 8.10 (Action of a magnetic operation on Bloch momentum)**.** *Let ψ be a generalized Bloch state with translation character $t_R\psi = e^{-ik\cdot R}\psi, (R \in L)$. Let $D(m)$ be the unitary or antiunitary operator representing*

$$m = \{p_m|\tau_m\}^\epsilon \in M.$$

Then $D(m)\psi$ has translation character $R \longmapsto e^{-i(\eta(m)p_m k)\cdot R}$. Equivalently, $D(m) : \mathcal{H}_k \longrightarrow \mathcal{H}_{\eta(m)p_m k}$.

Proof. The spatial conjugation rule is $m^{-1}t_R m = t_{p_m^{-1}R}$. Hence

$$t_R D(m)\psi = D(m)\big(m^{-1}t_R m\big)\psi = D(m)t_{p_m^{-1}R}\psi.$$

Since ψ has momentum k, $t_{p_m^{-1}R}\psi = e^{-ik\cdot p_m^{-1}R}\psi$. If m is unprimed, then $D(m)$ is linear, so

$$t_R D(m)\psi = e^{-ik\cdot p_m^{-1}R}D(m)\psi = e^{-i(p_m k)\cdot R}D(m)\psi.$$

If m is primed, then $D(m)$ is antilinear, so the scalar is complex conjugated:

$$t_R D(m)\psi = e^{+ik\cdot p_m^{-1}R}D(m)\psi = e^{-i(-p_m k)\cdot R}D(m)\psi.$$

Combining the two cases gives

$$t_R D(m)\psi = e^{-i(\eta(m)p_m k)\cdot R}D(m)\psi.$$

□

Definition 8.23. *The magnetic little co-group of k is the image of M_k after dividing out the translation subgroup T_0. Equivalently, it consists of the unitary or antiunitary point operations that fix k modulo reciprocal lattice vectors:*

$$\overline{M}_k = \{(\eta, p) : \eta p k = k \mod L^*\}.$$

A small corepresentation at k is a corepresentation D_k of M_k such that translations act by the Bloch character: $D_k(t_R) = e^{-ik\cdot R}I$.

Thus the classification of Bloch states at a magnetic high-symmetry momentum proceeds in two steps. First, the unitary translations fix the momentum k. Second, the remaining finite magnetic little co-group acts on the finite-dimensional space of states at that momentum. Because this little co-group may contain antiunitary elements, the appropriate labels are corepresentations, not just ordinary irreducible representations.

Proposition 8.11. *The magnetic little group fits into an exact sequence*

$$1 \longrightarrow T_0 \longrightarrow M_k \longrightarrow \overline{M}_k \longrightarrow 1.$$

On a Bloch fiber $\mathcal{H}_k$, the subgroup T_0 acts by the scalar character $t_R \longmapsto e^{-ik\cdot R}$. Therefore all nontrivial finite-dimensional symmetry labels at fixed k come from corepresentations of the magnetic little co-group, possibly with factor systems coming from spin and fractional translations.

Proof. The kernel of the map from M_k to the magnetic little co-group is the subgroup of operations whose point part is the identity and whose antiunitary label is trivial. These are precisely the unitary translations T_0. The map is onto by definition of $\overline{M}_k$. Hence the sequence is exact.
On the Bloch fiber, every vector satisfies $t_R\psi = e^{-ik\cdot R}\psi$. Thus T_0 acts by scalars, and the remaining matrix structure is carried by the quotient M_k/T_0. □

Fractional translations in magnetic space groups produce momentum dependent phases, just as in ordinary nonsymmorphic space groups. The only new feature is that some of the relevant operations may be antiunitary.

Definition 8.24. *Choose representatives $s(\alpha) = \{p_\alpha|\tau_\alpha\}^{\epsilon_\alpha}$ for elements α of a magnetic little co-group. If*

$$s(\alpha)s(\beta) = t_{R(\alpha,\beta)}s(\alpha\beta),$$

then $R(\alpha, \beta) \in L$, and on the Bloch fiber

$$D_k(s(\alpha))D_k(s(\beta)) = e^{-ik\cdot R(\alpha,\beta)}D_k(s(\alpha\beta)).$$

The phase $\omega_k(\alpha, \beta) = e^{-ik\cdot R(\alpha,\beta)}$ is the nonsymmorphic factor system at momentum k.

For spinful particles one must multiply this translation factor by the spin factor system. Thus, in general,

$$\omega_k(\alpha, \beta) = \omega_{\text{spin}}(\alpha, \beta)e^{-ik\cdot R(\alpha,\beta)}.$$

This compact formula is a useful way to remember where band degeneracies can come from: a minus sign may arise either from spin, from a Brillouin-zone boundary phase, or from both.

Theorem 8.9 (General antiunitary degeneracy criterion)**.** *Let A be an antiunitary symmetry of a Hamiltonian H, and suppose A preserves a Hilbert subspace, such as a Bloch fiber $\mathcal{H}_k$. Assume $AH = HA$ on this subspace and $A^2 = \lambda I$ with $\lambda \in U(1)$. If $\lambda \neq 1$, then every eigenstate ψ in this subspace has an orthogonal partner $A\psi$ with the same energy.*

Proof. If $H\psi = E\psi$, then $H(A\psi) = A(H\psi) = A(E\psi) = E^*A\psi$. Since H is Hermitian, E is real, so $H(A\psi) = E(A\psi)$. Thus $A\psi$ has the same energy as ψ.Let $x = \langle A\psi|\psi\rangle$. Using antiunitarity,

$$\langle A\psi|A(A\psi)\rangle = \langle A\psi|\psi\rangle = x.$$

On the other hand, since $A^2 = \lambda I$,

$$\langle A\psi|A(A\psi)\rangle = \langle A\psi|\lambda\psi\rangle = \lambda\langle A\psi|\psi\rangle = \lambda x.$$

Therefore $x = \lambda x$. If $\lambda \neq 1$, then $x = 0$. Hence $\langle A\psi|\psi\rangle = 0$. Thus ψ and $A\psi$ are orthogonal and therefore linearly independent. □

The usual Kramers theorem is the special case $\lambda = -1$. The more general version is useful in crystals because an antiunitary symmetry may square not to $\pm I$ abstractly, but to a translation, a screw, a glide, or a spin rotation. On a Bloch fiber, such an operation can become a scalar phase depending on k.

Theorem 8.10 (Kramers degeneracy)**.** *Let A be an antiunitary symmetry of a Hamiltonian H: $AH = HA$. If $A^2 = -I$, then every energy eigenstate has a linearly independent partner with the same energy. More precisely, if $H\psi = E\psi$, then $H(A\psi) = E(A\psi)$ and $\langle\psi|A\psi\rangle = 0$.*

Proof. This is the previous theorem with $\lambda = -1$. Equivalently, one can prove it directly. Since A commutes with H, $H(A\psi) = A(H\psi) = A(E\psi) = E(A\psi)$, because E is real. Thus $A\psi$ has the same energy. Let $y = \langle\psi|A\psi\rangle$. Then $y^* = \langle A\psi|\psi\rangle$. Using the general argument above with $\lambda = -1$, we get $\langle A\psi|\psi\rangle = 0$. Therefore $y = 0$. Thus ψ and $A\psi$ are orthogonal and linearly independent. □

It is important that the antiunitary symmetry preserve the same Hilbert space in which the degeneracy is being claimed. If time reversal sends $\mathcal{H}_k$ to $\mathcal{H}_{-k}$ and $k \neq -k \mod L^*$, then it relates the spectrum at k to the spectrum at $-k$. It does not force a twofold degeneracy at the same k. Degeneracy at a fixed momentum occurs when the antiunitary operation belongs to the magnetic little group of that momentum.

Example 8.7 (Bloch Hamiltonian constraints from time reversal)**.** *Let $H(k)$ be a Bloch Hamiltonian. For spinless time reversal, one may choose a basis in which $\Theta = K$. The time-reversal condition is $\Theta H(k)\Theta^{-1} = H(-k)$, or equivalently $H(k)^* = H(-k)$. At a momentum satisfying $k = -k \mod L^*$, this says that $H(k)$ can be represented by a real Hermitian matrix, hence by a real symmetric matrix. A real symmetric matrix need not have degenerate eigenvalues, so spinless time reversal alone does not imply Kramers degeneracy.*
For spinful electrons,

$$\Theta = i\sigma_y K, \qquad \Theta^2 = -1.$$

The time-reversal constraint becomes

$$(i\sigma_y)H(k)^*(-i\sigma_y) = H(-k),$$

with additional identity factors on orbital and sublattice degrees of freedom. At a time-reversal-invariant momentum, this antiunitary symmetry preserves the same Bloch fiber and squares to -1. Therefore every energy level at that momentum is at least twofold degenerate.

Definition 8.25. *For ordinary time reversal, the point part is $p = I$ and $\eta = -1$. Therefore $k \longmapsto -k$. A momentum satisfying $k = -k \mod L^*$ is called a time-reversal-invariant momentum, or TRIM.*

If

$$L^* = \mathbb{Z}b_1 \oplus \cdots \oplus \mathbb{Z}b_d,$$

then the TRIMs are

$$k_\nu = \frac{1}{2}\left(\nu_1 b_1 + \cdots + \nu_d b_d\right), \qquad \nu_i \in \{0, 1\}.$$

Thus there are 2^d TRIMs in d dimensions.

Example 8.8 (TRIMs in one dimension). *Let $L = a\mathbb{Z}$. Then $L^* = \frac{2\pi}{a}\mathbb{Z}$. The condition $k = -k \mod L^*$ means $2k = \frac{2\pi n}{a}$ for some integer n. Modulo L^*, the two solutions are $k = 0, k = \frac{\pi}{a}$. For spinful electrons, each energy level at these two momenta is at least twofold degenerate if time reversal is a symmetry.*

Example 8.9 (Combined inversion and time reversal). *Let P be inversion, so its point part is $p_P = -I$. Time reversal has $\eta(\Theta) = -1$. Therefore the combined antiunitary operation $P\Theta$ acts on momentum by $k \longmapsto (-1)(-I)k = k$. Thus $P\Theta$ belongs to the magnetic little group at every momentum k.*
If the system is spinful, P usually satisfies $P^2 = 1$ and commutes with Θ. Hence $(P\Theta)^2 = P^2\Theta^2 = -1$. Therefore a spinful system with both inversion and time reversal has Kramers degeneracy at every k, not only at the TRIMs.
If the system is spinless, then $(P\Theta)^2 = +1$. In that case there is no Kramers degeneracy forced by $P\Theta$, although the Bloch Hamiltonian can often be chosen real in a suitable basis.

Example 8.10 (Antiferromagnetic translation in one dimension). *Consider a one-dimensional collinear antiferromagnet with moments alternating up and down:*

$$\uparrow \downarrow \uparrow \downarrow \;\cdots.$$

Let the spacing between neighboring sites be a. Translation by a is not a symmetry of the magnetic structure because it exchanges up and down moments. However, translation by a followed by time reversal is a symmetry: $A = 1't_a$. The unitary translation subgroup is generated by t_{2a}. Thus the magnetic translation lattice is $L = 2a\mathbb{Z}$, and the magnetic reciprocal lattice is $L^* = \frac{\pi}{a}\mathbb{Z}$.
For a spinless magnetic structure,

$$A^2 = (1't_a)(1't_a) = t_{2a}.$$

Since A is antiunitary, it sends $k \longmapsto -k$. Therefore A belongs to the magnetic little group at the momenta satisfying

$$k = -k \mod \frac{\pi}{a}\mathbb{Z}.$$

In the reduced magnetic Brillouin zone, these are

$$k = 0, \qquad k = \frac{\pi}{2a}.$$

On the k-fiber,

$$D_k(A)^2 = D_k(t_{2a}) = e^{-i2ak}.$$

At $k = 0$, one has $D_k(A)^2 = +1$, so no degeneracy is forced by A alone. At the zone boundary $k = \frac{\pi}{2a}$, one has

$$D_k(A)^2 = e^{-i\pi} = -1.$$

Therefore the generalized Kramers criterion gives a twofold degeneracy even in a spinless description.
For spinful electrons, one must multiply by the spin factor $\Theta^2 = -1$. In that case the same antiunitary translation has $D_k(A)^2 = -e^{-i2ak}$. The momenta at which $D_k(A)^2 \neq 1$ are correspondingly shifted. This is a simple example of how spin phases and magnetic translation phases combine.

Example 8.11 (Glide-time and screw-time operations). *Antiunitary nonsymmorphic symmetries are common in magnetic space groups. Suppose an antiunitary operation A satisfies $A^2 = t_R$ in the spinless crystallographic group. On a Bloch fiber, $D_k(A)^2 = e^{-ik\cdot R}$. If $e^{-ik\cdot R} \neq 1$, then A forces a degeneracy at that k, provided A belongs to the magnetic little group.*
This is the antiunitary analogue of ordinary nonsymmorphic band sticking. In the ordinary unitary case, glide and screw eigenvalues vary with k. In the magnetic antiunitary case, the square of the antiunitary operation can itself become a nontrivial Bloch phase, producing a Kramers-like degeneracy even when the microscopic spin part has $\Theta^2 = +1$.

We can summarize the band-theoretic role of a magnetic group as follows. The unitary translation subgroup first decomposes the Hilbert space into Bloch fibers:

$$\mathcal{H} \cong \int_{\mathbb{R}^d/L^*}^{\oplus} \mathcal{H}_k \, dk.$$

A unitary magnetic symmetry maps $\mathcal{H}_k \longrightarrow \mathcal{H}_{pk}$. An antiunitary magnetic symmetry maps $\mathcal{H}_k \longrightarrow \mathcal{H}_{-pk}$. Only the symmetries that return k to itself modulo L^* act inside the single fiber $\mathcal{H}_k$. These symmetries form the magnetic little group M_k, and their finite-dimensional symmetry labels are corepresentations.
If a Hamiltonian H has magnetic symmetry M, then

$$D(m)HD(m)^{-1} = H$$

for all $m \in M$. In momentum space this implies

$$H(\eta(m)p_m k) \cong H(k).$$

Thus magnetic symmetry relates bands not only at point-group-related momenta pk, but also at momenta $-pk$ when the operation is antiunitary. At momenta fixed by an antiunitary symmetry, the square of that antiunitary symmetry determines whether a Kramers-type degeneracy is enforced.

Exercise 8.9. *Let Θ be spinless time reversal, so $\Theta^2 = 1$. Does Kramers degeneracy follow from Θ alone? Identify the precise step in the proof of Kramers degeneracy that fails.*

Exercise 8.10. *Let M_x be the mirror in two dimensions with point part*

$$p_{M_x}(k_x, k_y) = (-k_x, k_y).$$

Let $A = \Theta M_x$. Show that A acts on momentum by

$$(k_x, k_y) \longmapsto (k_x, -k_y).$$

For a rectangular lattice, determine the lines in the Brillouin zone on which A belongs to the magnetic little group.

Exercise 8.11. *Let A be an antiunitary symmetry preserving a Bloch fiber $\mathcal{H}_k$, and suppose $A^2 = e^{i\phi}I$. Prove that if $e^{i\phi} \neq 1$, then every eigenstate of an A-symmetric Hamiltonian has a degenerate orthogonal partner.*

Exercise 8.12. *In the one-dimensional antiferromagnetic translation example, verify explicitly that the magnetic Brillouin zone is*

$$\mathbb{R}/\frac{\pi}{a}\mathbb{Z}.$$

Then show that the antiunitary translation $A = 1't_a$ belongs to the magnetic little group only at $k = 0$, and $k = \frac{\pi}{2a}$. Compute $D_k(A)^2$ at both momenta.

Exercise 8.13. *Let P be inversion and Θ spinful time reversal. Assume*

$$P^2 = 1, \qquad P\Theta = \Theta P, \qquad \Theta^2 = -1.$$

Show that $(P\Theta)^2 = -1$. Explain why this forces a twofold degeneracy at every k, rather than only at time-reversal-invariant momenta.

Exercise 8.14. *Let $M = M_0 \sqcup aM_0$ be a finite magnetic group and let ρ be an irreducible representation of M_0. Define*

$$\rho^a(h) = \rho(a^{-1}ha)^*.$$

Explain physically the difference between the cases

$$\rho^a \simeq \rho \quad \text{and} \quad \rho^a \not\simeq \rho.$$

In the second case, why should one expect the antiunitary symmetry to pair two different unitary representation labels?

8.4 The $\vec{k} \cdot \vec{p}$ expansion

The $\vec{k} \cdot \vec{p}$ method constructs a local effective Hamiltonian near a chosen momentum k_0 in the Brillouin zone. The basic idea is simple: instead of trying to understand the full Bloch Hamiltonian $H(k)$ on the whole Brillouin zone, one writes $k = k_0 + q$ and expands in powers of the small deviation q. The coefficients of this expansion are not arbitrary. They are constrained by the symmetries that keep k_0 fixed, namely by the little group of k_0.
In physics, the $\vec{k} \cdot \vec{p}$ method has two closely related meanings. First, it is a perturbation method for deriving an effective Hamiltonian from microscopic Bloch wavefunctions. Second, it is a symmetry method for writing the most general local Hamiltonian allowed near a high-symmetry momentum. In this subsection we focus mainly on the second viewpoint, because it is the one most directly controlled by group representation theory. The perturbative derivation explains where the name "$\vec{k} \cdot \vec{p}$" comes from.
Let k_0 be a momentum in the Brillouin zone, and let $q = k - k_0$ be a small displacement. We assume that k_0 is a high-symmetry momentum, or more generally a point, line, or plane with a nontrivial little group. Let G_{k_0} denote the little group of k_0. Thus an element $g \in G_{k_0}$ maps k_0 to itself modulo a reciprocal lattice vector: $gk_0 = k_0 \mod L^*$. If g has point part p_g, then near k_0 it sends the deviation q to $q \longmapsto p_g q$.
If antiunitary symmetries are present, the same formula must be modified by the sign coming from complex conjugation. If a is antiunitary with point part p_a, then it acts on momentum as

$$k \longmapsto -p_a k.$$

Thus, if a belongs to the magnetic little group of k_0, then near k_0 it acts on the small momentum as

$$q \longmapsto -p_a q.$$

This is the local version of the magnetic little-group action discussed earlier.

Definition 8.26. *Let E be a finite-dimensional vector space spanned by a chosen group of bands at k_0. The space E is often called the local band space, the active subspace, or the low-energy subspace. Suppose E carries a representation, or corepresentation if antiunitary symmetries are present,*

$$D : G_{k_0} \to GL(E).$$

A $\vec{k} \cdot \vec{p}$ Hamiltonian near k_0 is a matrix-valued function $H_{\text{eff}}(q) \in \text{End}(E)$ expanded as a polynomial, or formal power series, in the components of q:

$$H_{\text{eff}}(q) = H^{(0)} + H^{(1)}(q) + H^{(2)}(q) + \cdots ,$$

where $H^{(n)}(\lambda q) = \lambda^n H^{(n)}(q)$.

Physically, E is the set of states we want to keep. For example, E might be a single nondegenerate band, a two-dimensional irreducible representation at Γ, a Kramers pair at a time-reversal-invariant momentum, or a four-dimensional Dirac multiplet. The rest of the bands are called remote bands. Their effect is hidden in the numerical coefficients of the effective Hamiltonian.
The effective Hamiltonian must be Hermitian: $H_{\text{eff}}(q)^\dagger = H_{\text{eff}}(q)$. Therefore, in practice, one writes it as a real linear combination of Hermitian matrices:

$$H_{\text{eff}}(q) = \sum_\alpha f_\alpha(q) M_\alpha,$$

where $M_\alpha^\dagger = M_\alpha$, and the scalar functions $f_\alpha(q)$ are real-valued for real q. In a two-band problem, the matrices M_α are usually chosen to be $I, \sigma_x, \sigma_y, \sigma_z$. In a larger problem, one may use matrix units, Gell-Mann matrices, angular momentum matrices, or any other convenient Hermitian basis.

Proposition 8.12 (Symmetry constraints on a local Hamiltonian)**.** *Let $g \in G_{k_0}$ be a unitary symmetry with point part p_g. Then the local Hamiltonian must satisfy*

$$D(g) H_{\text{eff}}(q) D(g)^{-1} = H_{\text{eff}}(p_g q).$$

If $a \in G_{k_0}$ is an antiunitary symmetry represented on E by $D(a) = U_a K$, where K denotes complex conjugation, then the local Hamiltonian must satisfy

$$U_a H_{\text{eff}}(q)^* U_a^{-1} = H_{\text{eff}}(-p_a q).$$

Proof. A unitary symmetry g maps the Bloch fiber at $k_0 + q$ to the Bloch fiber at $g(k_0 + q) = gk_0 + p_g q$. Since $gk_0 = k_0 \mod L^*$, this is the same local expansion point with displacement $p_g q$. If the symmetry is represented on the active subspace by $D(g)$, then applying g before or after the Hamiltonian must give the same physical result. Hence

$$D(g)H_{\text{eff}}(q)D(g)^{-1} = H_{\text{eff}}(p_g q).$$

For an antiunitary symmetry a, the momentum transforms as $k \longmapsto -p_a k$. Thus, near k_0, $q \longmapsto -p_a q$. Writing $D(a) = U_a K$, the action of $D(a)$ on a matrix is

$$D(a)H_{\text{eff}}(q)D(a)^{-1} = U_a H_{\text{eff}}(q)^* U_a^{-1}.$$

Therefore the symmetry constraint becomes $U_a H_{\text{eff}}(q)^* U_a^{-1} = H_{\text{eff}}(-p_a q)$. □

These two equations are the working rules of the method. They say that each allowed matrix term must transform in the same way as the polynomial in q multiplying it. If a polynomial and a matrix transform differently, their product cannot appear in the Hamiltonian.

One should be slightly careful at Brillouin-zone boundary points, especially for nonsymmorphic space groups. If a little-group operation contains a fractional translation, then its representation on the Bloch fiber contains a phase such as $e^{-ik_0 \cdot \tau}$. This phase is already included in the little-group representation $D(g)$ at k_0. In a fully microscopic construction, the sewing matrix may also have q-dependent phases. For the standard local method of invariants, one usually fixes the representation at k_0 and builds all polynomials consistent with that representation. If the q-dependent sewing phases are important for a particular nonsymmorphic model, they should be expanded consistently order by order.

Where the name $\vec{k} \cdot \vec{p}$ comes from

Consider a continuum single-particle Hamiltonian

$$H = \frac{p^2}{2m} + V(x), \qquad p = -i\hbar\nabla,$$

where the potential is periodic: $V(x + R) = V(x), (R \in L)$. A Bloch wave has the form $\psi_{n,k}(x) = e^{ik\cdot x}u_{n,k}(x)$, where $u_{n,k}(x+R) = u_{n,k}(x)$. Conjugating H by the plane-wave factor gives the cell-periodic Bloch Hamiltonian

$$H(k) = e^{-ik\cdot x}He^{ik\cdot x} = \frac{(p + \hbar k)^2}{2m} + V(x).$$

Proposition 8.13 (Basic $\vec{k} \cdot \vec{p}$ expansion)**.** *Let $k = k_0 + q$. Then*

$$H(k_0 + q) = H(k_0) + \frac{\hbar}{m}q \cdot (p + \hbar k_0) + \frac{\hbar^2 q^2}{2m}.$$

In particular, if $k_0 = 0$, then

$$H(q) = H(0) + \frac{\hbar}{m}q \cdot p + \frac{\hbar^2 q^2}{2m}.$$

The linear term is the origin of the name $\vec{k} \cdot \vec{p}$.

Proof. Starting from

$$H(k) = \frac{(p + \hbar k)^2}{2m} + V(x),$$

substitute $k = k_0 + q$. Then $p + \hbar k = p + \hbar k_0 + \hbar q$. Therefore

$$(p + \hbar k)^2 = (p + \hbar k_0)^2 + 2\hbar q \cdot (p + \hbar k_0) + \hbar^2 q^2.$$

Dividing by $2m$ gives

$$H(k_0 + q) = H(k_0) + \frac{\hbar}{m}q \cdot (p + \hbar k_0) + \frac{\hbar^2 q^2}{2m}.$$

□

If $u_1, \dots, u_N$ are cell-periodic eigenfunctions at k_0 spanning the active subspace E, then the matrix elements of the linear term are

$$\left(H^{(1)}_{\text{eff}}(q)\right)_{mn} = \frac{\hbar}{m}\sum_i q_i \langle u_m|(p_i + \hbar k_{0,i})|u_n\rangle.$$

These are the usual momentum matrix elements. Symmetry often forces many of them to vanish.

If the active bands are coupled to remote bands, one can integrate out the remote bands using degenerate perturbation theory or Löwdin partitioning. Let $\mathcal{P}$ be the projection onto the active subspace and let $\mathcal{Q} = 1 - \mathcal{P}$ project onto remote bands. A schematic second-order effective Hamiltonian is

$$H_{\text{eff}}(E, q) = H_{\mathcal{PP}}(q) + H_{\mathcal{PQ}}(q) \frac{1}{E - H_{\mathcal{QQ}}(q)} H_{\mathcal{QP}}(q) + \cdots .$$

The denominators contain energy differences to remote bands, while the numerators contain matrix elements of the $\vec{k} \cdot \vec{p}$ perturbation. This procedure determines the numerical coefficients in a model. Group theory, by contrast, determines which coefficients are allowed to be nonzero.
In applications, one often writes the most general symmetry-allowed Hamiltonian directly, with undetermined real constants. These constants are then obtained from experiment, from first-principles calculations, or from a microscopic tight-binding model. This is the method of invariants used in semiconductor and topological-band theory [14, 12, 4, 19].

Invariant theory formulation

Let Q be the vector representation carried by the small momentum q. Concretely, if q has component $q_1, \ldots, q_d$, then a group element g acts by $q \longmapsto p_g q$. Linear functions of q transform in the dual representation $Q^\vee$. Homogeneous polynomials of degree n transform as $\mathrm{Sym}^n(Q^\vee)$.
The matrices acting on the active space E transform by conjugation:

$$M \longmapsto D(g) M D(g)^{-1}.$$

As a representation, the space of matrices is

$$\mathrm{End}(E) \cong E^\vee \otimes E.$$

Therefore an nth-order term in the Hamiltonian is an element of

$$\mathrm{Sym}^n(Q^\vee) \otimes E^\vee \otimes E.$$

It is symmetry-allowed precisely when it is invariant under the little group.

Theorem 8.11 (Counting unitary invariants). *Suppose G_{k_0} is an ordinary finite unitary little group. The number of independent complex symmetry-allowed terms of degree n is*

$$N_n = \dim \mathrm{Hom}_{G_{k_0}} \left(1, \mathrm{Sym}^n(Q^\vee) \otimes E^\vee \otimes E\right).$$

Equivalently,

$$N_n = \left\langle 1, \chi_{\mathrm{Sym}^n(Q^\vee)} \chi_{E^\vee \otimes E} \right\rangle_{G_{k_0}},$$

where

$$\langle \chi, \psi \rangle_{G_{k_0}} = \frac{1}{|G_{k_0}|} \sum_{g \in G_{k_0}} \chi(g)^* \psi(g).$$

Proof. An nth-order Hamiltonian term is a tensor in $\mathrm{Sym}^n(Q^\vee) \otimes E^\vee \otimes E$. The symmetry condition says exactly that this tensor is fixed by every element of G_{k_0}. Thus the space of allowed terms is the invariant subspace

$$\left(\mathrm{Sym}^n(Q^\vee) \otimes E^\vee \otimes E\right)^{G_{k_0}}.$$

The dimension of this invariant subspace is the multiplicity of the trivial representation.
For a finite group, the multiplicity of an irreducible representation with character χ_α inside a representation with character χ is $\langle \chi_\alpha, \chi \rangle$. Taking $\chi_\alpha = 1$ and using the fact that characters multiply under tensor products gives

$$N_n = \left\langle 1, \chi_{\mathrm{Sym}^n(Q^\vee)} \chi_{E^\vee \otimes E} \right\rangle_{G_{k_0}}.$$

□

The formula above counts complex invariant tensors. A physical Hamiltonian is Hermitian, so in practice one chooses a Hermitian matrix basis and real coefficient functions. If a complex invariant appears together with its complex conjugate, the two are combined into real and imaginary Hermitian terms. This is the same distinction familiar from writing

$$v q_+ \sigma_- + v^* q_- \sigma_+$$

rather than treating the two complex monomials independently.

The character formula is efficient for counting terms, but it does not automatically write them down. To construct the terms explicitly, one decomposes the polynomial functions and the matrix space into irreducible representations. A product $f(q)M$ is invariant when the polynomial $f(q)$ and the matrix M transform in dual representations. For real point-group representations, the dual representation is often equivalent to the same representation.

Proposition 8.14 (Projection operator for constructing invariants)**.** *Let $F(q)$ be any matrix-valued homogeneous polynomial of degree n. For an ordinary unitary little group G_{k_0}, define*

$$(\Pi F)(q) = \frac{1}{|G_{k_0}|} \sum_{g \in G_{k_0}} D(g) F(p_g^{-1} q) D(g)^{-1}.$$

Then ΠF satisfies

$$D(h)(\Pi F)(q) D(h)^{-1} = (\Pi F)(p_h q)$$

for all $h \in G_{k_0}$. Thus ΠF is a symmetry-allowed term.

Proof. Compute

$$D(h)(\Pi F)(q) D(h)^{-1} = \frac{1}{|G_{k_0}|} \sum_{g \in G_{k_0}} D(hg) F(p_g^{-1} q) D(hg)^{-1}.$$

Set $r = hg$. Then $g = h^{-1} r$ and hence $p_g^{-1} = p_r^{-1} p_h$. Therefore

$$D(h)(\Pi F)(q) D(h)^{-1} = \frac{1}{|G_{k_0}|} \sum_{r \in G_{k_0}} D(r) F(p_r^{-1} p_h q) D(r)^{-1}.$$

This is exactly $(\Pi F)(p_h q)$. □

If antiunitary symmetries are present, a practical procedure is the following. First construct all terms allowed by the unitary subgroup. Then impose each antiunitary constraint

$$U_a H(q)^* U_a^{-1} = H(-p_a q).$$

This second step may force some coefficients to vanish, or may force some coefficients to be real or imaginary. This is often easier than trying to use characters of a magnetic group directly.

Schur's lemma and the zeroth-order term

The zeroth-order Hamiltonian is $H^{(0)} = H_{\text{eff}}(0)$. It must commute with the representation of the little group: $D(g) H^{(0)} D(g)^{-1} = H^{(0)}$. If E is an irreducible complex representation of the unitary little group, then Schur's lemma gives $H^{(0)} = E_0 I$. Thus all states in E are degenerate at $q = 0$ before symmetry-breaking perturbations are included.

If E is reducible, then $H^{(0)}$ is block diagonal with one block for each irreducible representation. Distinct irreducible representations are not forced to have the same energy. Accidental degeneracies between different irreducible representations can occur, but they are not protected by the little group alone.

This is a useful physical rule. A degeneracy at a high-symmetry point is symmetry-enforced when the states form a multidimensional irreducible representation, or when antiunitary symmetry produces a Kramers-type corepresentation. Once the degeneracy is identified, the $\vec{k} \cdot \vec{p}$ Hamiltonian describes how the degeneracy splits as one moves away from the high-symmetry momentum.

A practical recipe

The following recipe is often the fastest way to build a local model.

First, choose the momentum k_0 and determine the little group G_{k_0}. If antiunitary symmetries are present, use the magnetic little group. Remember that an antiunitary operation sends $q \longmapsto -pq$.

Second, choose the active band space E This means choosing the set of bands retained in the effective model. Determine the representation matrices $D(g)$ on this space. For spinful electrons, $D(g)$ should include the spin rotation. For nonsymmorphic operations, $D(g)$ should include the Bloch phase at k_0.

Third, choose a Hermitian basis of matrices on E. For a two-band model this is usually $I, \sigma_x, \sigma_y, \sigma_z$. For a four-band model it may be useful to use tensor products $\tau_i \sigma_j$, where τ_i acts on orbital or sublattice degrees of freedom and σ_j acts on spin.

Fourth, decompose the polynomial functions of q into irreducible representations of the little group. For example, in two dimensions one often uses

$$q_\pm = q_x \pm i q_y.$$

Under a rotation by angle θ,

$$q_\pm \longmapsto e^{\pm i\theta} q_\pm.$$

Fifth, pair polynomial functions with matrices transforming in the same representation so that the product is a scalar under the little group. Finally, impose Hermiticity and any antiunitary constraints.

Angular momentum selection rule

For rotations, the invariant method can be written as a simple angular momentum selection rule. This form is especially useful for physicists because it immediately tells which power of q_+ or q_- can couple two states.

Proposition 8.15 (Cyclic selection rule)**.** *Let C_n be generated by an n-fold rotation. Suppose two basis states $|\alpha\rangle, |\beta\rangle$ have C_n eigenvalues*

$$C_n|\alpha\rangle = e^{2\pi i j_\alpha/n}|\alpha\rangle, \qquad C_n|\beta\rangle = e^{2\pi i j_\beta/n}|\beta\rangle.$$

Let $M_{\alpha\beta} = |\alpha\rangle\langle\beta|$. Then $M_{\alpha\beta}$ transforms with character $e^{2\pi i(j_\alpha - j_\beta)/n}$. The monomial $q_+^r q_-^s$ transforms with character $e^{2\pi i(r-s)/n}$. Therefore a coupling $q_+^r q_-^s M_{\alpha\beta}$ is allowed by C_n if and only if

$$r - s \equiv j_\alpha - j_\beta \mod n.$$

Proof. Under C_n,

$$D(C_n) M_{\alpha\beta} D(C_n)^{-1} = e^{2\pi i j_\alpha/n} e^{-2\pi i j_\beta/n} M_{\alpha\beta} = e^{2\pi i(j_\alpha - j_\beta)/n} M_{\alpha\beta}.$$

We also have

$$q_+ \mapsto e^{2\pi i/n} q_+, \qquad q_- \mapsto e^{-2\pi i/n} q_-.$$

Hence $q_+^r q_-^s \mapsto e^{2\pi i(r-s)/n} q_+^r q_-^s$. The symmetry condition requires the polynomial and the matrix to transform in the same way, so

$$e^{2\pi i(r-s)/n} = e^{2\pi i(j_\alpha - j_\beta)/n}.$$

This is equivalent to

$$r - s \equiv j_\alpha - j_\beta \mod n.$$

□

This rule is the local version of angular momentum conservation modulo a reciprocal-lattice rotation. It explains, for instance, why some band crossings are linear, while others are quadratic or cubic. The difference is often just the angular momentum mismatch between the two states.

Example 8.12 (A D_4 doublet at Γ)**.** *Consider a spinless two-dimensional system with D_4 symmetry at $\Gamma = (0,0)$. Assume the two states at Γ transform as the two-dimensional vector representation E of D_4, with a basis similar to (p_x, p_y). The small momentum $q = (q_x, q_y)$ also transforms as the same representation E.*
We use the standard conjugacy classes of D_4:

$$\{e\}, \quad \{C_4, C_4^{-1}\}, \quad \{C_2\}, \quad \{m_x, m_y\}, \quad \{m_d, m_{d'}\}.$$

The character table is

	e	$2C_4$	C_2	$2m_v$	$2m_d$
A_1	1	1	1	1	1
A_2	1	1	1	−1	−1
B_1	1	−1	1	1	−1
B_2	1	−1	1	−1	1
E	2	0	−2	0	0

where m_v denotes mirrors along the coordinate axes and m_d denotes diagonal mirrors.
The matrix space is

$$\mathrm{End}(E) \cong E^\vee \otimes E.$$

Since E is real, $E^\vee \cong E$. The character of $E \otimes E$ is the square of the character of E:

$$\chi_{E\otimes E} = (4, 0, 4, 0, 0).$$

Taking inner products with the irreducible characters gives

$$E \otimes E = A_1 \oplus A_2 \oplus B_1 \oplus B_2.$$

A convenient Hermitian matrix basis is $I, \sigma_x, \sigma_y, \sigma_z$. With the usual vector-representation basis, these matrices transform as

$$I \sim A_1, \qquad \sigma_y \sim A_2, \qquad \sigma_z \sim B_1, \qquad \sigma_x \sim B_2.$$

Here σ_y represents the antisymmetric matrix structure. Although σ_y is imaginary as a Hermitian matrix, it transforms under D_4 in the same way as the real antisymmetric tensor

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Linear functions of q transform as $(q_x, q_y) \sim E$. Since the matrix space contains only A_1, A_2, B_1, B_2 and contains no copy of E, there is no first-order invariant. Therefore no linear splitting is allowed by D_4 in this two-dimensional representation.

Quadratic functions decompose as

$$\mathrm{Sym}^2(E) = A_1 \oplus B_1 \oplus B_2.$$

A symmetry-adapted basis is

$$q_x^2 + q_y^2 \sim A_1,$$
$$q_x^2 - q_y^2 \sim B_1,$$
$$2q_xq_y \sim B_2.$$

Pairing these polynomials with matrices of the same symmetry gives the most general second-order Hamiltonian

$$H_{\text{eff}}(q) = E_0 I + a(q_x^2 + q_y^2)I + b(q_x^2 - q_y^2)\sigma_z + c(2q_xq_y)\sigma_x.$$

Here $E_0, a, b, c \in \mathbb{R}$.

The two eigenvalues are

$$E_\pm(q) = E_0 + a(q_x^2 + q_y^2) \pm \sqrt{b^2(q_x^2 - q_y^2)^2 + 4c^2q_x^2q_y^2}.$$

Thus the degeneracy at Γ splits quadratically rather than linearly. In this minimal model, the result is a symmetry-protected quadratic band touching.

The A_2 matrix σ_y does not appear at second order because there is no quadratic A_2 polynomial. The first D_4 polynomial transforming as A_2 is

$$q_xq_y(q_x^2 - q_y^2) \sim A_2,$$

which is fourth order. Therefore, if time reversal is not imposed, a fourth order term

$$\lambda q_xq_y(q_x^2 - q_y^2)\sigma_y$$

is allowed by D_4. If spinless time reversal K is also a symmetry, this term is forbidden because $K\sigma_yK^{-1} = -\sigma_y$ while the fourth-order polynomial is even under $q \mapsto -q$.

Example 8.13 (Time reversal and a single Kramers pair)**.** *Consider a single Kramers pair near a time-reversal-invariant momentum k_0. The effective Hamiltonian has the form*

$$H_{\text{eff}}(q) = \epsilon(q)I + d_x(q)\sigma_x + d_y(q)\sigma_y + d_z(q)\sigma_z.$$

Spinful time reversal is represented by

$$\Theta = i\sigma_y K, \qquad \Theta^2 = -1.$$

The time-reversal constraint is

$$\Theta H_{\text{eff}}(q)\Theta^{-1} = H_{\text{eff}}(-q).$$

Since $\Theta I\Theta^{-1} = I$ and $\Theta\sigma_i\Theta^{-1} = -\sigma_i, (i = x, y, z)$, we obtain $\epsilon(q) = \epsilon(-q)$, and $d_i(q) = -d_i(-q)$. Thus the scalar part of the Hamiltonian is even in q, while the spin-splitting terms are odd in q.

To lowest order this gives

$$\epsilon(q) = \epsilon_0 + \sum_{i,j} A_{ij}q_iq_j + \cdots,$$
$$d_i(q) = \sum_j v_{ij}q_j + \cdots.$$

Time reversal therefore allows linear spin-orbit terms, because both spin and momentum change sign under time reversal.

If inversion symmetry is also present and acts trivially on this two-dimensional Kramers space, then inversion gives $H_{\text{eff}}(q) = H_{\text{eff}}(-q)$. This implies $d_i(q) = d_i(-q)$. Combining this with time reversal gives $d_i(q) = -d_i(q)$, so $d_i(q) = 0$. Therefore a centrosymmetric time-reversal-invariant crystal has no spin splitting of an isolated Kramers pair. This is the local $\vec{k} \cdot \vec{p}$ version of the familiar statement that inversion plus spinful time reversal forces double degeneracy at every k.

Example 8.14 (Rashba term as an invariant). *Consider a two-dimensional spinful system without inversion symmetry. Suppose the point group allows rotations in the plane, or at least enough rotational symmetry that* q_x, q_y *transform as an in-plane vector. The spin components* σ_x, σ_y *transform as components of an axial vector. The combination* $q_y\sigma_x - q_x\sigma_y$ *is a scalar under in-plane rotations. It is also even under time reversal, because time reversal sends* $q \mapsto -q$ *and* $\sigma \mapsto -\sigma$*. Therefore the linear spin-orbit term*

$$H_R(q) = \alpha(q_y\sigma_x - q_x\sigma_y)$$

is allowed by time reversal and by the rotational symmetry.
Under inversion, however, $q \mapsto -q$ *while spin, being an axial vector, does not change sign. Hence* $q_y\sigma_x - q_x\sigma_y$ *changes sign under inversion. Therefore the Rashba term is forbidden in an inversion-symmetric crystal or heterostructure. This is a simple example of how the method of invariants reproduces a standard spin-orbit Hamiltonian.*

Example 8.15 (A two-band mirror constraint). *Let a two-band model have a unitary mirror symmetry* M *represented by* $D(M) = \sigma_z$*. Suppose the mirror acts on momentum by* $(q_x, q_y) \longmapsto (q_x, -q_y)$*. Write the most general Hermitian two-band Hamiltonian as*

$$H(q_x, q_y) = h_0(q_x, q_y)I + h_x(q_x, q_y)\sigma_x + h_y(q_x, q_y)\sigma_y + h_z(q_x, q_y)\sigma_z.$$

The mirror constraint is

$$\sigma_z H(q_x, q_y)\sigma_z = H(q_x, -q_y).$$

Using $\sigma_z I \sigma_z = I, \sigma_z\sigma_z\sigma_z = \sigma_z$*, and* $\sigma_z\sigma_x\sigma_z = -\sigma_x, \sigma_z\sigma_y\sigma_z = -\sigma_y$*, we find*

$$\begin{aligned} h_0(q_x, q_y) &= h_0(q_x, -q_y), \\ h_x(q_x, q_y) &= -h_x(q_x, -q_y), \\ h_y(q_x, q_y) &= -h_y(q_x, -q_y). \\ h_z(q_x, q_y) &= h_z(q_x, -q_y). \end{aligned}$$

Thus h_0 *and* h_z *must be even in* q_y*, while* h_x *and* h_y *must be odd in* q_y*.*
Up to first order in q*, the most general mirror-symmetric Hamiltonian is therefore*

$$H(q_x, q_y) = (\epsilon_0 + v_0 q_x)I + (m + v_z q_x)\sigma_z + v_x q_y \sigma_x + v_y q_y \sigma_y,$$

with real coefficients. The terms proportional to $q_y I$ *and* $q_y\sigma_z$ *are forbidden, while the terms proportional to* $q_y\sigma_x$ *and* $q_y\sigma_y$ *are allowed.*

Example 8.16 (A C_3 selection rule). *Let* C_3 *act on momentum by rotation through* $2\pi/3$*. Define*

$$q_+ = q_x + iq_y, \qquad q_- = q_x - iq_y.$$

Then $q_+ \mapsto \omega q_+, q_- \mapsto \omega^{-1} q_-$*, where* $\omega = e^{2\pi i/3}$*.*
Suppose one band at Γ *has* C_3 *eigenvalue* ω*, and a second band has* C_3 *eigenvalue* 1*. Use the basis* $|1\rangle = |\omega\rangle, |2\rangle = |1\rangle$*. The off-diagonal matrix* $|1\rangle\langle 2|$ *transforms as* ω*. Since* q_+ *also transforms as* ω*, the linear coupling* $vq_+|1\rangle\langle 2|$ *is allowed. Hermiticity then requires the conjugate term* $v^* q_-|2\rangle\langle 1|$*. Thus the off-diagonal part of the Hamiltonian can contain*

$$H_{\text{off}}(q) = \begin{pmatrix} 0 & vq_+ \\ v^* q_- & 0 \end{pmatrix}.$$

In angular momentum language, the two states differ by one unit of C_3 *angular momentum. A single power of* q_+ *carries exactly this angular momentum, so a linear coupling is allowed. If the angular momentum mismatch were two units modulo* 3*, then* q_- *would be the allowed linear factor instead. If no linear monomial carried the required mismatch, the leading coupling would begin at quadratic or higher order.*

Antiunitary symmetries in local models

Antiunitary symmetries are especially important in $\vec{k} \cdot \vec{p}$ theory because they constrain the parity of terms in q. For ordinary time reversal, $q \mapsto -q$. For spinless time reversal, $\Theta = K$, so the constraint is $H(q)^* = H(-q)$. For spinful time reversal, $\Theta = i\sigma_y K$, so the constraint is $i\sigma_y H(q)^*(-i\sigma_y) = H(-q)$.
These equations should only be imposed inside a single local Hamiltonian when time reversal belongs to the little group of k_0. This means $k_0 = -k_0 \mod L^*$. If k_0 is not time-reversal invariant, then time reversal relates the $\vec{k} \cdot \vec{p}$ Hamiltonian near k_0 to a different $\vec{k} \cdot \vec{p}$ Hamiltonian near $-k_0$. It does not impose a constraint within one local model by itself.
More generally, if an antiunitary symmetry a maps $k_0 \longmapsto k_1$ with $k_1 \neq k_0 \mod L^*$, then it relates the expansion near k_0 to the expansion near k_1. Only when $k_1 = k_0 \mod L^*$ does a act inside the same local band space E and impose a constraint of the form $U_a H(q)^* U_a^{-1} = H(-p_a q)$.

Proposition 8.16 (Kramers constraint in a $\vec{k}\cdot\vec{p}$ model)**.** *Suppose an antiunitary symmetry A belongs to the little group of k_0 and satisfies $A^2 = -I$ on the active space E. Then $H_{\text{eff}}(0)$ has even degeneracy. More generally, if A preserves a line or plane in q-space and satisfies $A^2 = -I$ on that subspace, then the spectrum is at least twofold degenerate along that line or plane.*

Proof. At a momentum where A maps the Bloch fiber to itself, the operator A commutes with the Hamiltonian on that fiber:

$$AH_{\text{eff}}(q) = H_{\text{eff}}(q)A.$$

If $H_{\text{eff}}(q)\psi = E\psi$, then $H_{\text{eff}}(q)(A\psi) = A(H_{\text{eff}}(q)\psi) = E(A\psi)$, because E is real. Thus $A\psi$ is another eigenstate with the same energy.
Since $A^2 = -I$, the usual Kramers argument gives $\langle\psi|A\psi\rangle = 0$. Hence ψ and $A\psi$ are linearly independent. Therefore the eigenvalue is at least twofold degenerate. □

In nonsymmorphic or magnetic groups, an antiunitary operation may square to a translation: $A^2 = t_R$. On a Bloch fiber this becomes $D_k(A)^2 = e^{-ik\cdot R}$. Thus at a particular k_0, one may have $D_{k_0}(A)^2 = -1$ even for a spinless system. In that case the same Kramers argument applies to the local $\vec{k}\cdot\vec{p}$ Hamiltonian at k_0.

Relation to band crossings

A two-band Hamiltonian can be written as

$$H(q) = h_0(q)I + \vec{h}(q)\cdot\vec{\sigma}.$$

The two eigenvalues are

$$E_\pm(q) = h_0(q) \pm |\vec{h}(q)|.$$

Thus a band crossing occurs when $\vec{h}(q) = 0$. Symmetry controls which components of $\vec{h}$ are allowed and at what order in q they appear.
If all three Pauli matrices are allowed with independent linear coefficients in three dimensions, then a generic isolated crossing is a Weyl point. If only two Pauli matrices appear linearly in two dimensions, one obtains a Dirac cone. If symmetry forbids all linear terms, as in the D_4 doublet example above, the leading splitting may be quadratic. Thus the order of the band touching is often a direct representation-theoretic consequence of the little group.
The identity matrix I does not split a degeneracy. It only shifts both energies equally. Therefore, when diagnosing band crossings, one usually focuses on the non-scalar matrices. The scalar terms determine the tilt, curvature, and particle-hole asymmetry of the bands, but not the symmetry-protected degeneracy itself.

Summary of the method

The $\vec{k}\cdot\vec{p}$ construction can be summarized in one line:

$$\text{allowed terms} = (\text{polynomials in } q) \otimes (\text{matrices on } E) \quad \text{that contain the trivial representation.}$$

For ordinary unitary little groups, this is encoded by

$$N_n = \left\langle 1, \chi_{\mathrm{Sym}^n(Q^\vee)}\chi_{E^\vee\otimes E}\right\rangle_{G_{k_0}}.$$

For magnetic little groups, one first builds the terms allowed by the unitary subgroup and then imposes the antiunitary constraints explicitly.
This viewpoint is powerful because it separates symmetry from microscopic details. Symmetry determines the form of the Hamiltonian. The microscopic physics determines the numerical coefficients. Two materials with the same little-group representation at k_0 have the same allowed $\vec{k}\cdot\vec{p}$ terms, even if their coefficients are very different.

Exercise 8.15. *For the D_4 doublet example, use the character table of D_4 to verify*

$$E\otimes E = A_1 \oplus A_2 \oplus B_1 \oplus B_2.$$

Then use $E\otimes A_i \cong E, E\otimes B_i \cong E$, to explain why no first-order invariant can be formed from a linear function of q and a matrix in $\mathrm{End}(E)$.

Exercise 8.16. *In the D_4 doublet example, show that $q_xq_y(q_x^2-q_y^2)$ transforms as A_2. Conclude that $q_xq_y(q_x^2-q_y^2)\sigma_y$ is allowed by D_4. Then impose spinless time reversal K and determine whether this term remains allowed.*

Exercise 8.17. *Let a two-band model near* k_0 *have a unitary mirror symmetry* M *represented by* $D(M) = \sigma_z$*, and suppose the mirror sends* $(q_x, q_y) \mapsto (q_x, -q_y)$*. Find all terms up to first order in* q *allowed by* $\sigma_z H(q_x, q_y)\sigma_z = H(q_x, -q_y)$*. Then repeat the calculation after imposing spinless time reversal* $KH(q_x, q_y)K^{-1} = H(-q_x, -q_y)$*.*

Exercise 8.18. *Let* C_3 *act on momentum by rotation through* $2\pi/3$*. Suppose a one-dimensional band at* Γ *has* C_3 *eigenvalue* $\omega = e^{2\pi i/3}$*, and another one-dimensional band has eigenvalue 1. Use the angular momentum selection rule to determine whether a linear coupling is allowed. Write the allowed off-diagonal term explicitly using* $q_\pm = q_x \pm i q_y$*.*

Exercise 8.19. *Let two states have* C_4 *eigenvalues* $e^{2\pi i j_1/4}$ *and* $e^{2\pi i j_2/4}$*. Find the lowest possible order of a coupling between them in terms of the congruence class* $j_1 - j_2 \mod 4$*. Apply your answer to the cases* $j_1 - j_2 = 1, 2, 3 \mod 4$*.*

Exercise 8.20. *Consider a single spinful Kramers pair at a time-reversal-invariant momentum. Starting from*

$$H(q) = \epsilon(q)I + \sum_i d_i(q)\sigma_i,$$

use $\Theta = i\sigma_y K$ *to prove*

$$\epsilon(q) = \epsilon(-q), \qquad d_i(q) = -d_i(-q).$$

Then impose inversion symmetry acting as the identity on the Kramers pair and show that all $d_i(q)$ *vanish.*

Exercise 8.21. *Let* $H(q) = v_x q_x \sigma_x + v_y q_y \sigma_y + m\sigma_z$ *be a two-dimensional Dirac Hamiltonian. Determine which of the three terms are allowed if the system has a mirror symmetry represented by* σ_z *and acting as* $(q_x, q_y) \mapsto (q_x, -q_y)$*. What additional constraint is imposed by spinless time reversal?*

Exercise 8.22. *Let* G *be a finite unitary little group and let* $F(q)$ *be a matrix-valued polynomial. Verify that*

$$(\Pi F)(q) = \frac{1}{|G|}\sum_{g\in G} D(g)F(p_g^{-1}q)D(g)^{-1}$$

satisfies the symmetry condition

$$D(h)(\Pi F)(q)D(h)^{-1} = (\Pi F)(p_h q).$$

Use this projection operator to construct one allowed quadratic term in the D_4 *doublet example.*

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